ISSN: 0976-3031

Available Online at http://www.recentscientific.com

CODEN: IJRSFP (USA)

International Journal of Recent Scientific Research
Vol. 9, Issue, 5(H), pp. 27016-27021, May, 2018

International Journal of Recent Scientific Research

DOI: 10.24327/IJRSR

Research Article

A GENERAL QUEUING MODEL WITHARBITRARY SERVICE TIME DISTRIBUTION

Aparna Kushwaha[1]., Anshula Pandey[2] and Varun Kumar Kashyap*[3]

[1]Department of Statistics, University of Lucknow, Lucknow-226007 (INDIA)
[2]NMIMS, Mumbai
[3]Jamiya Hamdard Institute of Medical Sciences and Research New Delhi-110060

DOI: http://dx.doi.org/10.24327/ijrsr.2018.0905.2169

ARTICLE INFO

Article History:
Received 10th February, 2018
Received in revised form 6th March, 2018
Accepted 24th April, 2018
Published online 28th May, 2018

Key Words:

Queueing system, arrival distribution, service time distribution (Erlang, Lomax and Laplace distribution)

ABSTRACT

This paper deals with queuing system when arrival distribution is Poisson, service time distribution is arbitrary with mean service rate μ per unit of time, there is single service channel, system capacity is unlimited and service discipline is first come first serve(FCFS). The steady state equations governing the queue are obtained. Many characteristics for the model viz probability density function (p. d. f.) for waiting time distribution, busy period distribution, etc. are obtained. Particular cases when the service time follows Erlang, Lomax and Laplace distributions, are considered and for all the three cases the steady state equations and various characteristics for the model are derived.

INTRODUCTION

We are availing several service facilities in our daily life routine, for eg. at a service station we may encounter situations and service related tasks such Customers waiting for check-out service in a supermarket/complex mall, Cars waiting for parking or at a stop light on a road crossing, Flights waiting for take-off or landing at an airport and Damaged machines waiting for repairing services. These situations have been very common phenomenon of waiting. The Pioneer of Queuing theory was Danish Mathematician A.K. Erlang (1909) who published "The Theory of Probability". The prodigious mathematician A.K. Erlang known as father of 'Teletraffic Theory' obtained the formulae related to traffic loads. The mathematical discussion on queuing theory substantially progressed in early 1930,s through the work of Pollaczk (1930, 1934), Kolmogorov (1931), Khintchine (1932, 1955), and others. Kendall (1951, 1953) gave a symmetric treatment of stochastic process occur the theory of Queues and Cox (1955) analyzed the congestion problems statistically. khintchine (1960) discussed mathematical methods in Queueing theory. Morse (1958) discussed many different kind of special Queuing problems in a wide way and Lee, A.M. (1958) gave applied queuing theory. An element of queuing theory with applications was given by T.L. Saaty (1961). Solution of certain typical problems in queuing and scheduling theory has been given by Arpana Badoni (2001).

Steady state equations for the model

Let $P_n(t)$ be the probability of n units in the system at time t and $P_n(t+\Delta t)$ be the probability that the system has n units at time $(t+\Delta t)$.

The event can occur in the following mutually exclusive and exhaustive ways.

1. There are n units in the system at time t and there is no arrival, no service completion during the time interval Δt.
2. There are $(n-1)$ units in the system at time t and one arrival but no service completion at time interval Δt.
3. There are $(n+1)$ units in the system at time t and no arrival but one service completion at time interval Δt.
4. There are n units in the system at time t and no arrival and no service completion at time interval Δt.

Thus for $n \geq 1$;

Corresponding author:* **Varun Kumar Kashyap
Jamiya Hamdard Institute of Medical Sciences and Research New Delhi-110060

$P_n(t+\Delta t) = P_n(t).P(\text{no arrival at } \Delta t).P(\text{no service at } \Delta t) +$
$P_{n-1}(t).P(1 \text{ arrival at } \Delta t).P(\text{no service at } \Delta t) +$
$P_{n+1}(t).\ P(\text{no arrival at } \Delta t).P(1 \text{ service at } \Delta t) + P_n(t).\ P(1 \text{ arrival at } \Delta t).$
$P(1 \text{ service at } \Delta t) + O(\Delta t)$

[Since arrivals and services occur in randomly independent way and probability of more than 1 arrival or 1 service is negligible in small time interval Δt].

$P_n(t+\Delta t) = P_n(t)\{(1-\lambda)\Delta t + O(\Delta t)\}\{(1-\mu)\Delta t + O(\Delta t)\} +$
$P_{n-1}(t)\{\lambda\Delta t + O(\Delta t)\}\{(1-\mu)\Delta t + O(\Delta t)\}+$
$P_{n+1}(t)\{(1 - \lambda t) + O(\lambda\Delta t)\}\{\mu\lambda t + O(\Delta t)\} + O(\Delta t)$; $n \geq 1$

Or, $P_n(t+\Delta t) = P_n(t) - (\lambda+\mu) P_n(t)\,\Delta t + \lambda P_{n-1}(t)\,\Delta t + \mu P_{n+1}(t)\Delta t + O(\Delta t)$; $n \geq 1$

{Combining all the terms of O(Δt)}

Or, $\lim_{\Delta t \to 0} \frac{P_n(t+\Delta t)-P_n(t)}{\Delta t} = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t) + \lim_{\Delta t \to 0} \frac{O(\Delta t)}{\Delta t}$
; $n \geq 1$

Or, $P_n'(t) = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t)$; $n \geq 1$ ….(2.1)

(Since $\lim_{\Delta t \to 0} \frac{O(\Delta t)}{\Delta t} \to 0$)
For n = 0, equation (2.1) is not valid.
For n = 0, the event that there is no unit in the system at time (t+Δt), occurs in two mutually exclusive and exhaustive ways, i.e.

1. There is no unit in the system at time'‘t’ and no arrival during ‘Δt’ (as the system is empty thus any question service doesn’t arise).Or
2. There is one unit in the system at time‘t’ and during ‘Δt’ there is no arrival but one service completion.

Thus, for n = 0,
$P_0(t+\Delta t) = P_0(t).P[\text{no arrival during } \Delta t] +$

$P_1(t).P[\text{no arrival during } \Delta t].P[1 \text{ service completion during } \Delta t] + O(\Delta t)$;(Since arrivals and services are independent)
$= P_0(t)\ \{(1-\lambda)\Delta t + O(\Delta t)\} + P_1(t)\{(1-\lambda)\Delta t + O(\Delta t\}.\{\mu\Delta t + O(\Delta t)\} + O(\Delta t)$
$= P_0(t) - \lambda P_0(t)\,\Delta t + \mu P_1(t)\Delta t + O(\Delta t)$ {Combining all the terms of O(Δt)}
$\lim_{\Delta t \to 0} \frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = -\lambda P_0(t) + \mu P_1(t) + \lim_{\Delta t \to 0} \frac{O(\Delta t)}{\Delta t}$

Or, $P_0'(t) = -\lambda P_0(t) + \mu P_1(t)$; since $\lim_{\Delta t \to 0} \frac{O(\Delta t)}{\Delta t} \to 0$ ….(2.2)

Under steady state conditions $P_n(t) \to P_n$ for large t so that $P_n'(t) \to 0 \quad \forall n$.

Thus equation (2.1) and (2.2) reduce to
$-(\lambda+\mu) P_n + \mu P_{n+1} + \lambda P_{n-1} = 0$
Or, $P_{n+1} = \frac{1}{\mu}[(\lambda+\mu)P_n - \lambda P_{n-1}]$ ….(2.3)
And, $\mu P_1 - \lambda P_0 = 0$
Therefore, $P_1 = \frac{\lambda}{\mu} P_0$ ….(2.4)

Putting n = 1 in equation (2.3) and using (2.4), we get
$P_2 = \frac{1}{\mu}[(\lambda+\mu) P_1 - \lambda P_0]$
$P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0$ …. (2.5)

Putting n = 2 in equation (2.3) and using (2.4), (2.5) we get,
$P_3 = \frac{1}{\mu}[(\lambda+\mu).P_2 - \lambda P_1]$
$= \left(\frac{\lambda}{\mu}\right)^3 P_0$

Let the result is true for n = r ; i.e.

$P_r = \left(\frac{\lambda}{\mu}\right)^r \left(\frac{\lambda}{\mu}\right)^r$ …. (2.6)

Thus from (2.3), we get

$P_{r+1} = \frac{1}{\mu}[(\lambda+\mu)P_r - \lambda P_{r-1}]$
$= \frac{1}{\mu}[(\lambda+\mu)\left(\frac{\lambda}{\mu}\right)^r P_0 - \lambda\left(\frac{\lambda}{\mu}\right)^{r-1} P_0] = \left(\frac{\lambda}{\mu}\right)^{r+1} P_0$
Hence the result is true for n = (r+1).

Since it is true for n = 1, 2… therefore by method of mathematical induction, it holds for all positive integral values of n i.e.

$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$; Since, $\sum_{n=0}^{\infty} P_n = 1$ …. (2.7)

Therefore, $P_0 \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n = 1 = P_0\left[\frac{1}{1-\frac{\lambda}{\mu}}\right] = 1$

$P_0 = \left(1 - \frac{\lambda}{\mu}\right)$ …. (2.8)

Hence the steady-state equation for the model is

$P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$ …. (2.9)

Characteristics of the model

Probability that queue length ≥ k

$P[\text{queue Size} \geq k] = \sum_{n=k}^{\infty} P_n$
$= \sum_{n=0}^{\infty} P_n - \sum_{n=0}^{k-1} P_n$
$= 1 - \sum_{n=0}^{k-1} \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$
$= 1 - \left(1 - \frac{\lambda}{\mu}\right) \cdot \sum_{n=0}^{k-1} \left(\frac{\lambda}{\mu}\right)^n$
$= 1 - \frac{(1-\lambda/\mu)\{1-(\lambda/\mu)^k\}}{(1-\lambda/\mu)}$ (Sum of k terms of G.P)

Thus, $P[\text{queue Size} \geq k] = \left(\frac{\lambda}{\mu}\right)^k$ …. (2.10)

Average number of customers in the system

The average number of customers in the system is given by
$E(n) = \sum_{n=0}^{\infty} nP_n$
$= \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$
$= \left(1 - \frac{\lambda}{\mu}\right) S$

Where, $S = \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n$ is the sum of miscellaneous series, whose first term is A.P. and second term is G.P. with common ratio $\frac{\lambda}{\mu}$
Or; $S = \frac{\lambda}{\mu} + 2\left(\frac{\lambda}{\mu}\right)^2 + 3\left(\frac{\lambda}{\mu}\right)^3 + \ldots\ldots\ldots\ldots\ldots$
$\frac{\lambda}{\mu} S = \left(\frac{\lambda}{\mu}\right)^2 + 2\left(\frac{\lambda}{\mu}\right)^3 + \ldots\ldots\ldots\ldots\ldots$
Subtracting, we get
$\left(1 - \frac{\lambda}{\mu}\right) S = \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3 + \ldots\ldots\ldots\ldots\ldots$

Or, $S = \frac{(\frac{\lambda}{\mu})}{[1-(\frac{\lambda}{\mu})]^2}$

Thus, the average number of customers in the system is,

$E(n) = \left(1 - \frac{\lambda}{\mu}\right) S$

$= \left(1 - \frac{\lambda}{\mu}\right) \frac{(\frac{\lambda}{\mu})}{[1-(\frac{\lambda}{\mu})]^2}$

Therefore, $E(n) = \frac{\lambda}{\mu-\lambda}$ .... (2.11)

Average queue length: Queue length is the length of queue excluding the person being served. Therefore, queue length is given by, E(m) where m= (n−1)

$L_q = E(m) = \sum mP_n$; m is a function of n

$=\sum_{n=1}^{\infty}(n-1)P_n$

$=\sum_{n=1}^{\infty} nP_n - \sum_{n=1}^{\infty} P_n$

$= \sum_{n=0}^{\infty} nP_n - \sum_{n=1}^{\infty} P_n - P_0 + P_0$

$= E(n) - \sum_{n=0}^{\infty} P_n + P_0$

$= \frac{(\frac{\lambda}{\mu})}{(1-\frac{\lambda}{\mu})} - 1 + \left(1 - \frac{\lambda}{\mu}\right)$

$= \frac{(\frac{\lambda}{\mu})}{(1-\frac{\lambda}{\mu})} - \frac{\lambda}{\mu}$

$= \left(\frac{\lambda}{\mu}\right)\left[\frac{1-1+(\frac{\lambda}{\mu})}{1-\frac{\lambda}{\mu}}\right] = \frac{(\frac{\lambda}{\mu})^2}{(1-\frac{\lambda}{\mu})}$

$L_q = E(m) = \frac{(\frac{\lambda}{\mu})^2}{(1-\frac{\lambda}{\mu})}$ …. (2.12)

Average length of non-empty queue

It is given by E[m/m>0] = $\frac{E(m)}{P(m>0)} = \frac{E(m)}{P(n-1>0)} = \frac{E(m)}{P(n>1)} = \frac{E(m)}{1-P(n\le 0)} = \frac{E(m)}{1-P(n\le 0)}$

$= \frac{E(m)}{1-P_0-P_1} = \frac{\frac{(\frac{\lambda}{\mu})^2}{(1-\frac{\lambda}{\mu})}}{1-\left(1-\frac{\lambda}{\mu}\right)-\left\{\frac{\lambda}{\mu}\left(1-\frac{\lambda}{\mu}\right)\right\}} = \frac{\frac{(\frac{\lambda}{\mu})^2}{(1-\frac{\lambda}{\mu})}}{\frac{\lambda}{\mu}-\frac{\lambda}{\mu}+(\frac{\lambda}{\mu})^2}$

$E[m/m>0] = \left(\frac{1}{1-\frac{\lambda}{\mu}}\right) = \frac{\mu}{\mu-\lambda}$ …. (2.13)

Variance of queue length is given by

$V(n) = E[n - E(n)]^2$

$= E[n^2] - \{E(n)\}^2$

$= E[n^2] - \left(\frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}}\right)^2$

$E(n^2) = \sum_{n=0}^{\infty} n^2 P_n$

$= \sum_{n=0}^{\infty}\{n(n-1) + n\}P_n$

$=\sum_{n=0}^{\infty} n(n-1)P_n + \sum_{n=0}^{\infty} nP_n$

$= \sum_{n=0}^{\infty} n(n-1)P_n + E(n)$

$= \sum_{n=0}^{\infty} n(n-1)\left(\frac{\lambda}{\mu}\right)^n\left\{1-\frac{\lambda}{\mu}\right\} + \left\{\frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}}\right\}$

$= \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^2 \sum_{n=0}^{\infty}\frac{d}{d\rho^2}\rho^n + \left\{\frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}}\right\}$;

where $\rho = \frac{\lambda}{\mu}$

$= \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^2 \frac{d}{d\rho^2}\sum_{n=0}^{\infty}\rho^n + \left\{\frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}}\right\}$

$= \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^2 \frac{d}{d\rho^2}\left(\frac{1}{1-\rho}\right) + \frac{\frac{\lambda}{\mu}}{\left(1-\frac{\lambda}{\mu}\right)}$

$= \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^2 \frac{2}{(1-\rho)^3} + \frac{\frac{\lambda}{\mu}}{\left(1-\frac{\lambda}{\mu}\right)}$

$= \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^2 \frac{2}{(1-\frac{\lambda}{\mu})^3} + \frac{\frac{\lambda}{\mu}}{\left(1-\frac{\lambda}{\mu}\right)}$

Or; $E(n^2) = \frac{2\left(\frac{\lambda}{\mu}\right)^2}{\left(1-\frac{\lambda}{\mu}\right)^2} + \frac{\left(\frac{\lambda}{\mu}\right)}{\left(1-\frac{\lambda}{\mu}\right)}$

Thus, $V(n) = E(n)^2 - \{E(n)\}^2$

$= \left\{\frac{2\left(\frac{\lambda}{\mu}\right)^2}{\left(1-\frac{\lambda}{\mu}\right)^2} + \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}}\right\} - \left\{\frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}}\right\}^2$

$= \left(\frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}}\right)^2 + \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}} = \frac{\frac{\lambda}{\mu}}{\left(1-\frac{\lambda}{\mu}\right)^2}\left[\frac{\lambda}{\mu} + 1 - \frac{\lambda}{\mu}\right.$

$V(n) = \frac{\frac{\lambda}{\mu}}{\left(1-\frac{\lambda}{\mu}\right)^2}$ ….... (2.14)

Probability density function of waiting time (excluding service time) distribution

In steady state, each customer has the same continuous waiting time distribution with probability density function φ(t). Let φ(t)dt is the probability that a customer begins to be served in the interval (t, t+dt), where t is measured from the time of his arrival.

Let a customer arrives at time t = 0 and service begins in the interval (t, t+Δt), then

1. If system is empty, then waiting time is zero with probability
   $P_0 = \left(1 - \frac{\lambda}{\mu}\right)$.
2. If there are n customers already in the system, when $(n+1)^{th}$ customer arrives, n customers must leave the system before service of $(n+1)^{th}$ customer begins **or** (n−1) customers must be served during the time interval (0, t) and n^{th} customer during (t, t+Δt).

As mean service rate is assumed to be μ per unit of time or μt in time t and probability of (n−1) departures in time t is given by Poisson distribution

$f(t) = \frac{(\mu t)^{n-1}e^{-\mu t}}{(n-1)!}$

If there are n customers in the system, then

$\varphi_n(t)$ = P[(n−1) customers are served by time t]×P[n^{th} customer is served during Δt]

$= \frac{(\mu t)^{n-1}e^{-\mu t}}{(n-1)!}.\mu\Delta t$

The probability density function of waiting time of a unit is given by

$= \sum_{n=1}^{\infty}\varphi_n(t).P_n$

$=\sum_{n=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right).\frac{(\mu t)^{n-1}e^{-\mu t}}{(n-1)!}\Delta t$

$= \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)\mu e^{-\mu t}.\sum_{n=1}^{\infty}\frac{(\mu t)^{n-1}}{(n-1)!}\left(\frac{\lambda}{\mu}\right)^{n-1}\Delta t$

$= \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right).\mu e^{-\mu t}\sum_{n=1}^{\infty}\frac{(\mu t)^{n-1}}{(n-1)!}\left(\frac{\lambda}{\mu}\right)^{n-1}\Delta t$

$=\lambda\left(1-\frac{\lambda}{\mu}\right)e^{-\mu t}[1+\lambda t+\frac{(\lambda t)^2}{2!}+\cdots]\Delta t$

$\varphi_n(t) = \lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)t}.\ \Delta t;\ t>0$ ....(2.15)

Now, $\int_0^\infty \varphi_n dt + P_0 = \lambda\left(1-\frac{\lambda}{\mu}\right)\int_0^\infty e^{-(\mu-\lambda)t}\,dt + \left(1-\frac{\lambda}{\mu}\right)$

$= \left(\frac{\lambda}{\mu}+1-\frac{\lambda}{\mu}\right) = 1$

Thus, the complete distribution of waiting time is partly continuous and partly discrete i.e.

1. It is continuous for $(t, t+\Delta t)$ with probability density function $=\frac{\lambda}{\mu}(\mu-\lambda)e^{-(\mu-\lambda)t}=\lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)t}$
2. Discrete for $t = 0$ with probability function $P_0=\left(1-\frac{\lambda}{\mu}\right)$

The probability that waiting time exceeds t is given by

$$P[W\geq t] = \int_t^\infty \varphi_{w(t)dt}$$

$$=\lambda\left(1-\frac{\lambda}{\mu}\right)\int_t^\infty e^{-(\mu-\lambda)t}\,dt$$

$$=\frac{\lambda}{\mu}(\mu-\lambda)\int_t^\infty e^{-(\mu-\lambda)t}\,dt$$

$$=\frac{\lambda}{\mu}e^{-(\mu-\lambda)t} \quad \text{.... (2.16)}$$

Probability distribution of time spent in the system (i.e. busy period distribution)

Let $\varphi_w(t/t>0)$ be the probability density function. For waiting time such that a person has to wait, i.e. the server remains busy in the busy period.

$$\varphi_w(t/t>0) = \frac{\varphi_w(t)}{P(w>0)}$$

$$=\frac{\varphi_w(t)}{\text{P[waiting time is greater than zero]}}$$

$$=\frac{\lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)t}}{\int_0^\infty \varphi_{w(t)dt}}=\frac{\lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)t}}{\int_0^\infty\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)t}}$$

$$==\frac{e^{-(\mu-\lambda)t}}{\int_0^\infty e^{-(\mu-\lambda)t}dt} = = (\mu-\lambda)e^{-(\mu-\lambda)t} \quad \text{.........(2.17)}$$

Expected time that an arrival spends in the system

It is the expected waiting time in the queue excluding service time & is given by

$$E(W) = \int_0^\infty t\varphi_w(t)dt = \lambda\left(1-\frac{\lambda}{\mu}\right)\int_0^\infty te^{-(\mu-\lambda)t}\,dt$$

$$=\lambda\left(1-\frac{\lambda}{\mu}\right)\left[t\,\frac{e^{-(\mu-\lambda)t}}{-(\mu-\lambda)}-\frac{1}{(\mu-\lambda)^2}e^{-(\mu-\lambda)t}\right]_0^\infty$$

$$=\lambda\left(\frac{\mu-\lambda}{\mu}\right).\frac{1}{(\mu-\lambda)^2}$$

$$E(W) = \frac{\lambda}{\mu(\mu-\lambda)} \quad \text{.... (2.18)}$$

Specific cases

Case I

Let the service time follows Erlang distribution with parameters (α,β) with probability density function (p.d.f.)

$$f(t) = \frac{\beta^\alpha}{(\alpha-1)!}t^{\alpha-1}e^{-\beta t}\ ;\ t\geq 0\ ;\ (\alpha,\beta)>0 \quad \text{....(2.1.1)}$$

With mean service rate,

$$E(t)=\int_0^\infty t.f(t)dt$$

$$=\int_0^\infty t.\frac{\beta^\alpha}{(\alpha-1)!}t^{\alpha-1}e^{-\beta t}dt$$

$$=\frac{\beta^\alpha}{(\alpha-1)!}\int_0^\infty\left(\frac{u}{\beta}\right)^{\alpha+1-1}e^{-\beta t}\frac{du}{\beta}$$

$$=\frac{1}{\beta(\alpha-1)!}\int_0^\infty u^{(\alpha+1)-1}\,e^{-u}du$$

$$=\frac{1}{\beta(\alpha-1)!}.\Gamma(\alpha+1)=\frac{\alpha\,\Gamma\alpha}{\beta(\alpha-1)!}\frac{\alpha(\alpha-1)!}{\beta(\alpha-1)!}$$

Therefore, for Erlang distribution, mean service rate $\mu=\frac{\alpha}{\beta}$

Putting $\mu=\frac{\alpha}{\beta}$ in (2.9), the Steady state equation for the model is

$$P_0=\left(1-\frac{\lambda\beta}{\alpha}\right) \text{ And } P_n=\left(\frac{\lambda\beta}{\alpha}\right)^n\left[1-\frac{\lambda\beta}{\alpha}\right] \quad \text{.... (2.1.3)}$$

Characteristics of the model

Probability that queue length ≥ k

$$P\text{ [queue size} \geq K] = \left(\frac{\lambda}{\mu}\right)^k = \left(\frac{\lambda\beta}{\alpha}\right)^k \quad \text{.... (2.1.4)}$$

***Average number of customers in the system** is given by*

$$E(n) = \left(\frac{\lambda}{\mu-\lambda}\right)=\frac{\lambda}{\left(\frac{\alpha}{\beta}\right)-\lambda}=\frac{\lambda\beta}{(\alpha-\beta\lambda)} \quad \text{.... (2.1.5)}$$

Average queue length is given by

$$E(m) = \frac{\left(\frac{\lambda\beta}{\alpha}\right)^2}{\left(1-\frac{\lambda\beta}{\alpha}\right)} \quad \text{.... (2.1.6)}$$

Average length of non-empty queue is given by

$$E[m/m>0]=\frac{\mu}{\mu-\lambda}=\frac{\frac{\alpha}{\beta}}{\frac{\alpha}{\beta}-\lambda}=\frac{\alpha}{\alpha-\beta\lambda} \quad \text{.... (2.1.7)}$$

Variance of queue length is given by

$$V(n)=\frac{\left(\frac{\lambda}{\mu}\right)}{\left(1-\frac{\lambda}{\mu}\right)^2}=\frac{\frac{\lambda\beta}{\alpha}}{\left(1-\frac{\lambda\beta}{\alpha}\right)^2} \quad \text{.... (2.1.8)}$$

Probability function of waiting time (excluding service time) is given by

$$\varphi_w(t)=\begin{cases}P_0 & ;\text{if system is empty}\\ \lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)t} & ;\text{otherwise}\end{cases}$$

$$=\begin{cases}\left(1-\frac{\lambda\beta}{\alpha}\right) & ;\text{if system is empty}\\ \lambda\left(1-\frac{\lambda\beta}{\alpha}\right)e^{-\left(\frac{\alpha}{\beta}-\lambda\right)t} & ;\text{otherwise}\end{cases} \quad \text{.... (2.1.9)}$$

Probability that waiting time exceeds time'*t*'

$$P[\text{waiting time} > t]=\frac{\lambda}{\mu}e^{-(\mu-\lambda)t}=\frac{\lambda\beta}{\alpha}e^{-\left(\frac{\alpha}{\beta}-\lambda\right)t} \quad \text{.... (2.1.10)}$$

Probability function of busy period is given by

$$=(\mu-\lambda)e^{-(\mu-\lambda)t}$$

$$=\left(\frac{\alpha}{\beta}-\lambda\right)e^{-\left(\frac{\alpha}{\beta}-\lambda\right)t} \quad \text{.... (2.1.11)}$$

Expected waiting time an arrival spends in the system (excluding service time) is given by;

$$E[W/W>0]=\frac{\lambda}{\mu(\mu-\lambda)}=\frac{\lambda}{\frac{\alpha}{\beta}\left(\frac{\alpha}{\beta}-\lambda\right)} \quad \text{.... (2.1.12)}$$

Case II

Let the service time follows Lomax distribution with parameters (α, β) i.e. the probability density function (p.d.f.) is given by;

$f(t) = \frac{\alpha}{\beta}\left[1+\frac{t}{\beta}\right]^{-(\alpha+1)} \quad ; t \geq 0 \quad ; (\alpha,\beta) > 0$ ….(2.2.1)

With mean service rate $\mu = E(t) = \int_{Rgt} t.f(t)dt$

$= \int_0^\infty t.\frac{\alpha}{\beta}\left[1+\frac{t}{\beta}\right]^{-(\alpha+1)} dt$

$= \frac{\alpha}{\beta}\int_0^\infty t.\frac{1}{\left[1+\frac{t}{\beta}\right]^{(\alpha+1)}} dt$

$= \frac{\alpha}{\beta}\int_0^\infty \frac{t^{2-1}}{\left[1+\frac{t}{\beta}\right]^{(\alpha+1)+2-2}} dt$

$= \frac{\alpha}{\beta}.\beta\int_0^\infty \frac{\left(\frac{t}{\beta}\right)^{2-1}}{\left[1+\frac{t}{\beta}\right]^{(\alpha+1)+2-2}} dt$

$= \alpha\int_0^\infty \frac{\left(\frac{t}{\beta}\right)^{2-1}}{\left[1+\frac{t}{\beta}\right]^{2+(\alpha-1)}} dt$

$\mu = E(u) = \alpha\int_0^\infty \frac{(u)^{2-1}}{[1+u]^{2+(\alpha-1)}}\beta.du$

$= \beta\alpha.\beta(2, \alpha-1)$

$\mu = \beta\alpha.\frac{\Gamma 2.\Gamma(\alpha-1)}{\Gamma(\alpha+1)}$

$\mu = \beta.\frac{\Gamma(\alpha-1)}{(\alpha-1)\Gamma(\alpha-1)}$

Thus, for Lomax distribution, mean service rate, $\mu = \frac{\beta}{\alpha-1}$; for $\alpha > 1$…. (2.2.2)

Putting $\mu = \frac{\beta}{\alpha-1}$ in equation (2.9), we get the steady state equation for the model when arrival distribution is Poisson, service distribution is Lomax, capacity of system is unlimited, there is single service channel and service discipline is FCFS (First come first served) basis is given by,

$P_0 = \left(1-\frac{\lambda}{\mu}\right) = 1-\frac{\lambda(\alpha-1)}{\beta}$

$P_n = \left(\frac{\lambda}{\mu}\right)^n\left[1-\frac{\lambda}{\mu}\right]$

$= \left(\frac{\lambda(\alpha-1)}{\beta}\right)^n\left[1-\frac{\lambda(\alpha-1)}{\beta}\right]$ …. (2.2.3)

Characteristics of the model

Probability that queue length ≥ k

$P[\text{queue size} \geq K] = \left(\frac{\lambda}{\mu}\right)^k = \left(\frac{\lambda(\alpha-1)}{\beta}\right)^k$ …. (2.2.4)

Average number of customers in the system is given by

$E[n] = \frac{\lambda}{\mu-\lambda} = \frac{\lambda}{\frac{\lambda(\alpha-1)}{\beta}-\lambda} = \frac{\lambda\beta}{\lambda(\alpha-1)-\lambda\beta}$ ….(2.2.5)

Average queue length is given by

$L_q = E(m) = \frac{\left(\frac{\lambda}{\mu}\right)^2}{1-\frac{\lambda}{\mu}} = \frac{\left(\frac{\lambda(\alpha-1)}{\beta}\right)^2}{1-\frac{\lambda(\alpha-1)}{\beta}}$

$E(m) = \frac{[\lambda(\alpha-1)]^2}{\beta[\beta-\lambda(\alpha-1)]}$ ….(2.2.6)

Average length of non-empty queue is given by

$E[m/m>0] = \frac{\mu}{\mu-\lambda} = \frac{\frac{\beta}{\alpha-1}}{\left(\frac{\beta}{\alpha-1}\right)-\lambda}$

$E[m/m>0] = \frac{\beta}{[\beta-\lambda(\alpha-1)]}$ ….(2.2.7)

Variance of queue length is given by

$V(n) = \frac{\frac{\lambda}{\mu}}{\left(1-\frac{\lambda}{\mu}\right)^2} = \frac{\left(\frac{\lambda(\alpha-1)}{\beta}\right)}{\left[1-\frac{\lambda(\alpha-1)}{\beta}\right]^2}$ …. (2.2.8)

Probability function of waiting time (excluding service time) is given by

$$\varphi_\omega(t) = \begin{cases} P_0 & ; \text{if system is empty} \\ \lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)t} & ; \text{otherwise} \end{cases}$$

$$= \begin{cases} 1-\frac{\lambda(\alpha-1)}{\beta} & ; \text{if system is empty} \\ \lambda\left(1-\frac{\lambda(\alpha-1)}{\beta}\right)e^{-\left(\frac{\beta}{\alpha-1}-\lambda\right)t} & ; \text{otherwise} \end{cases} \quad \text{…. (2.2.9)}$$

Probability that waiting time exceeds t

$P[\text{waiting time} > t] = \frac{\lambda}{\mu}e^{-(\mu-\lambda)t} = \frac{\lambda(\alpha-1)}{\beta}e^{-\left(\frac{\beta}{\alpha-1}-\lambda\right)t}$ …. (2.2.10)

Probability function of busy period is given by

$= (\mu-\lambda)e^{-(\mu-\lambda)t}$

$= \left(\frac{\beta}{\alpha-1}-\lambda\right)e^{-\left(\frac{\beta}{\alpha-1}-\lambda\right)t}$ …. (2.2.11)

Expected waiting time an arrival spent in the system (excluding service time) is given by

$E(W/W>0) = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\lambda}{\frac{\beta}{\alpha-1}\left\{\frac{\beta}{\alpha-1}-\lambda\right\}}$ …. (2.2.12)

Particularly, for α =1 all the above results are same when service time distribution is exponential.

Case III

Let the service time't' follows Laplace distribution with parameters (α, β) i.e. its probability density function (pdf) is

$f(t) = \frac{1}{2\beta}\exp\left(-\frac{|t-\alpha|}{\beta}\right); -\infty < t < \infty; \ (\alpha,\beta) > 0$ ….(2.3.1)

$$= \frac{1}{2\beta}\begin{cases} \exp\left(-\frac{\alpha-t}{\beta}\right); & \text{if } t < \mu \\ \exp\left(-\frac{t-\alpha}{\beta}\right); & \text{if } t \geq \mu \end{cases}$$

Here, $\alpha > 0$ is a location parameter and $\beta > 0$, is a scale parameter, which is sometimes referred to as the diversity.

Mean service rate, $\mu = E(t) = \int_{Rgt} t.f(t)dt$

$= E(u+\alpha)$

$= E(u)+\alpha$

$= \frac{1}{2\beta}\int_{-\infty}^{\infty} u\, e^{\frac{-|u|}{\beta}} du + \alpha$

$= \frac{1}{2\beta}\int_{-\infty}^{0} u\, e^{\frac{u}{\beta}} du + \frac{1}{2\beta}\int_0^\infty u\, e^{-\frac{u}{\beta}} du + \alpha$; putting u = −u

$= -\frac{1}{2\beta}\int_0^\infty u\, e^{-\frac{u}{\beta}} du + \frac{1}{2\beta}\int_0^\infty u\, e^{-\frac{u}{\beta}} du + \alpha$

Thus, for Lomax distribution, mean service rate, $\mu = \alpha$…. (2.3.2)

Putting $\mu = \alpha$ in equation (2.9), we get the steady state equation for the model when arrival distribution is Poisson, service distribution is Laplace, capacity of system is unlimited, there is single service channel and service discipline is FCFS (First come first served) basis is given by;

$P_0 = \left(1 - \frac{\lambda}{\mu}\right) = 1 - \frac{\lambda}{\alpha}$

$P_n = \left(\frac{\lambda}{\mu}\right)^n \left[1 - \frac{\lambda}{\mu}\right] = \left(\frac{\lambda}{\alpha}\right)^n \left[1 - \frac{\lambda}{\alpha}\right]$ …. (2.3.3)

Characteristics of the model

Probability that queue length ≥ k

P[queue size ≥ K] $= (\frac{\lambda}{\mu})^k$

$= \left(\frac{\lambda}{\alpha}\right)^k$ …. (2.3.4)

Average number of customers in the system is given by,

$E[n] = \frac{\lambda}{\mu-\lambda} = \frac{\lambda}{\alpha-\lambda} = \frac{\lambda}{\alpha-\lambda}$ …. (2.3.5)

Average queue length is given by

$L_q = E(m) = \frac{\left(\frac{\lambda}{\mu}\right)^2}{1-\frac{\lambda}{\mu}}$

$= \frac{\left(\frac{\lambda}{\alpha}\right)^2}{1-\frac{\lambda}{\alpha}} = \frac{\left(\frac{\lambda}{\alpha}\right)^2}{1-\frac{\lambda}{\alpha}} = \frac{[\lambda]^2}{\alpha[\alpha-\lambda]}$ …. (2.3.6)

Average length of non-empty queue is given by

$E[m/m>0] = \frac{\mu}{\mu-\lambda} = \frac{\alpha}{\alpha-\lambda}$ …. (2.3.7)

Variance of queue length is given by

$V(n) = \frac{\frac{\lambda}{\mu}}{\left(1-\frac{\lambda}{\mu}\right)^2} = \frac{\frac{\lambda}{\alpha}}{\left(1-\frac{\lambda}{\alpha}\right)^2}$ …. (2.3.8)

Probability function of waiting time (excluding service time) is given by

$$\varphi_\omega(t) = \begin{Bmatrix} P_0 & ;\text{if system is empty} \\ \lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)t} & ;\text{otherwise} \end{Bmatrix}$$

$$= \begin{Bmatrix} \left(1-\frac{\lambda}{\alpha}\right) & ;\text{if system is empty} \\ \lambda\left(1-\frac{\lambda}{\alpha}\right)e^{-(\alpha-\lambda)t} & ;\text{otherwise} \end{Bmatrix} \quad …. (2.3.9)$$

Probability that waiting time exceeds t

P[waiting time > t] $= \frac{\lambda}{\mu}e^{-(\mu-\lambda)t} = \frac{\lambda}{\alpha}e^{-(\alpha-\lambda)t}$ …. (2.3.10)

Probability function of busy period is given by

$= (\mu-\lambda)e^{-(\mu-\lambda)t}$

$= (\alpha-\lambda)e^{-(\alpha-\lambda)t}$ …. (2.3.11)

Expected waiting time an arrival spent in the system (excluding service time) is given by,

$E(W/W>0) = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\lambda}{\alpha(\alpha-\lambda)}$ …. (2.3.12)

From the above results, we find that characteristics are same when service time distribution is exponential.

References

1. Erlang, A.K. (1909), Probability and telephone calls, Nyt.Tidsskr.Mat., Ser.B. Vol.20, PP.33-39.
2. Kendal,D.G. (1951), Some problems in the theory of Queues, J. Roy.statist Soc.,Ser.B, Vol. 16, PP. 288-291
3. Kendal, D.G.(1953), Stochastic processes occurring in the theory of Queues and their analysis by the method of the imbedded Markov chain, *Ann. Math, Stats. Soc., Ser.* B, Vol. 16, PP. 338-354.
4. O' Brein, G.G. (1954), The solution of the Queueing Problems *J. Soc. Ind. Appl. Math*. Vol. 2 PP. 133-142.
5. Saaty, T.L. (1961), Elements of Queueing theory with applications, McGraw-Hill ook Company, New York.
6. Prabhu, N. U. & Bhatt, U. N., ((1962) Further result for the Queue with Poisson arrivals, Operation Research, Vol.11. PP. 380-386
7. Hiller, F.S. & Boiling, R. w.(1967), 0inite queues in Series with Exponential or Erlang ServiceTimes a numerical approach, Operation Research, Vol.15, PP. 286-303.
8. Arpana Badoni(2001), On some problems in queueing and scheduling system. Ph. D. thesis Garhwal University Srinagar.

How to cite this article:

Aparna Kushwaha *et al.*2018, A General Queuing Model Witharbitrary Service Time Distribution. *Int J Recent Sci Res.* 9(5), pp. 27016-27021. DOI: http://dx.doi.org/10.24327/ijrsr.2018.0905.2169

****