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Research Article

ON BI-QUADRATIC WITH FOUR UNKNOWNS $5xy+3z^2=3w^4$

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ABSTRACT

We obtain infinitely many non-zero integer quadruples (x, y, z, w) satisfying the bi-quadratic equation with four unknowns $5xy+3z^2=3w^4$.

INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, bi-quadratic diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity (Dickson 1952; Mordell 1969; Carmichael 1959; Telang 1996; Nigel D. Smart 1999). In this context, one may refer (Gopalan and Pandichelvi, 2009; Gopalan and Shanmuganandham, 2010; Gopalan and Sangeetha, 2010; Gopalan and Padma, 2010; Gopalan and Shanmuganandham, 2012; Gopalan and Sivakami, 2013; Gopalan *et al*, 2013; Vidhyalakshmi *et al*, 2014; Meena *et al*, 2014; Gopalan *et al*, 2015; Gopalan *et al*, 2015; Gopalan *et al*, 2016) for various problems on the bi-quadratic diophantine equations with four variables. However, often we come across non-homogeneous bi-quadratic equations and as such one may require its integral solution in its most general form. It is towards this end, this paper concerns with the problem of determining non-trivial integral solutions of the non-homogeneous equation with four unknowns given by $5xy+3z^2=3w^4$.

Method of Analysis

The bi-quadratic equation with four unknowns to be solved is given by

$$5xy+3z^2=3w^4 \tag{1}$$

Introduction of the linear transformations

$$x=u+v, \quad y=u-v, \quad z=v \tag{2}$$

in (1), leads to

$$5u^2-2v^2=3w^4 \tag{3}$$

Again, introducing the transformations

$$u=X+2T \ , \ v=X+5T \tag{4}$$

in (3), it gives

$$X^2=10T^2+w^4 \tag{5}$$

which is satisfied by

$$X=10m^2+n^2 \ , \ T=2mn \tag{6}$$

$$w^2=10m^2-n^2 \tag{7}$$

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Substituting the values of X and T from (6) in (4) and using (2), we have

$$\left.\begin{aligned} x &= 20m^2+2n^2+14mn \\ y &= -6mn \\ z &= 10m^2+n^2+10mn \end{aligned}\right\} \quad (8)$$

As our interest is on finding integer solutions to (1), to obtain w in integers from (7), assume

$$m = a^2+b^2 \quad (9)$$

Write 10 as

$$10 = (3+i)(3-i) \quad (10)$$

Using (9), (10) in (7) and employing the method of factorization, define

$$w+in = (3+i)(a+ib)^2$$

from which, we get

$$w = 3(a^2-b^2)-2ab \quad (11)$$

$$n = a^2-b^2+6ab \quad (12)$$

Substituting (9) and (12) in (8), we have

$$\left.\begin{aligned} x &= 36a^4+8b^4+108a^2b^2+108a^3b+60ab^3 \\ y &= -6a^4+6b^4-36a^3b-36ab^3 \\ z &= 21a^4+b^4+54a^2b^2+72a^3b+48ab^3 \end{aligned}\right\} \quad (13)$$

Thus, (11) and (13) represent the integer solutions to (1).

Note: 1

In addition to (10), one may also write 10 as

$$10 = (1+3i)(1-3i)$$

In this case, the corresponding values of x, y, z and w satisfying (1) are given by

$$x = 80a^4-4b^4+12a^2b^2+52a^3b+4ab^3$$
$$y = -18a^4+18b^4-12a^3b-12ab^3$$
$$z = 49a^4-11b^4+6a^2b^2+32a^3b+8ab^3$$
$$w = a^2-b^2-6ab$$

Remark: 1

The value of w from (7) may be obtained as follows:
Write (7) in the form of ratio as

$$\frac{w+3m}{m+n} = \frac{m-n}{w-3m} = \frac{\alpha}{\beta}, \quad \beta \neq 0$$

The above equation is equivalent to the system of double equations

$$\beta w+(3\beta-\alpha)m-\alpha n = 0$$

$$\alpha w-(\beta+3\alpha)m+\beta n = 0$$

Applying the method of cross multiplication, it is seen that

$$\left.\begin{aligned} m &= -(\alpha^2+\beta^2) \\ n &= \alpha^2-\beta^2-6\alpha\beta \end{aligned}\right\} \quad (14)$$

$$w = 3(\beta^2-\alpha^2)-2\alpha\beta \quad (15)$$

Substituting (14) in (8), we have

$$\left.\begin{aligned} x &= 8\alpha^4+36\beta^4+108\alpha^2\beta^2+60\alpha^3\beta+108\alpha\beta^3 \\ y &= 6\alpha^4-6\beta^4-36\alpha^3\beta-36\alpha\beta^3 \\ z &= \alpha^4+21\beta^4+54\alpha^2\beta^2+48\alpha^3\beta+72\alpha\beta^3 \end{aligned}\right\} \quad (16)$$

Thus, (15) and (16) represent the integer solutions to (1).

Note: 2

One may also write (7) in the form of ratio as

$$\frac{w+m}{3m+n} = \frac{3m-n}{w-m} = \frac{\alpha}{\beta}, \quad \beta \neq 0$$

Following the procedure presented above, the corresponding integer solutions to (1) are obtained as follows:

$$x = 4(-\alpha^4+20\beta^4+3\alpha^2\beta^2+\alpha^3\beta+13\alpha\beta^3)$$
$$y = 2(9\alpha^4-9\beta^4-6\alpha^3\beta-6\alpha\beta^3)$$
$$z = -11\alpha^4+49\beta^4+6\alpha^2\beta^2+8\alpha^3\beta+32\alpha\beta^3$$
$$w = \beta^2-\alpha^2-6\alpha\beta$$

Remark: 2

To solve (5), observe that it is written as the system of double equations as shown below in Table: 1

Table 1 System of double equations

System	1	2	3
$X+w^2$	$10T$	$5T$	$5T^2$
$X-w^2$	T	$2T$	2

Consider system: 1. On solving and employing (4), (2), we get

$$x = 36\alpha^2, \ y = -6\alpha^2, \ z = 21\alpha^2, \ w = \pm 3\alpha$$

In a similar manner, considering system: 2, we have

$$x = 84\alpha^2, \ y = -18\alpha^2, \ z = 51\alpha^2, \ w = \pm 3\alpha$$

Now, considering system: 3 and solving, we get

$$T = 2k, \ X = 10k^2+1 \quad (17)$$

$$w^2 = 10k^2-1 \quad (18)$$

From (17), (4) and (2), note that

$$\left.\begin{aligned} x &= 20k^2+14k+2 \\ y &= -6k \\ z &= 10k^2+10k+1 \end{aligned}\right\} \quad (19)$$

Now, to obtain w from (18), observe that it is the negative pellian equation whose smallest positive integer solution is

$$k_0 = 1\,,\ w_0 = 3$$

To obtain the other solutions of (18), consider its corresponding positive pell equation represented by

$$w^2 = 10k^2 + 1$$

whose general solution $\left(\tilde{k}_n, \tilde{w}_n\right)$ is given by

$$\tilde{w}_n = \frac{1}{2} f_n\,,\ \tilde{k}_n = \frac{1}{2\sqrt{10}} g_n$$

where

$$f_n = \left(19+6\sqrt{10}\right)^{n+1} + \left(19-6\sqrt{10}\right)^{n+1}\,,\ g_n = \left(19+6\sqrt{10}\right)^{n+1} - \left(19-6\sqrt{10}\right)^{n+1}\,,\ n = 0,1,2,....$$

Applying Brahmagupta lemma between (k_0, w_0) and $\left(\tilde{k}_n, \tilde{w}_n\right)$, the other integer solutions of (18) are given by

$$k_{n+1} = \frac{1}{2} f_n + \frac{3}{2\sqrt{10}} g_n \tag{20}$$

$$w_{n+1} = \frac{3}{2} f_n + \frac{\sqrt{10}}{2} g_n\ ,\ n = -1,0,1,2,.... \tag{21}$$

Substituting (20) in (19), we have

$$\left.\begin{aligned} x_{n+1} &= 20k_{n+1}^2 + 14k_{n+1} + 2 \\ y_{n+1} &= -6k_{n+1} \\ z_{n+1} &= 10k_{n+1}^2 + 10k_{n+1} + 1 \end{aligned}\right\} \tag{22}$$

Thus, (21) and (22) represent the integer solutions to (1).

Remark: 3

Introduction of the linear transformation

$$z = x - y \tag{23}$$

in (1) gives

$$3\left(x^2 + y^2\right) - xy = 3w^4 \tag{24}$$

Treating (24) as a quadratic in x and solving for x, we get

$$x = \frac{1}{6}\left(y \pm \sqrt{36w^4 - 35y^2}\right) \tag{25}$$

Let $\alpha^2 = 36w^4 - 35y^2$ (26)

Express (26) in the form of ratio as

$$\frac{\alpha + w^2}{w^2 + y} = \frac{35\left(w^2 - y\right)}{\alpha - w^2} = \frac{P}{Q},\quad Q > 0$$

Representing (26) as the system of double equations and employing the method of factorization, it is seen that

$$\alpha = P^2 + 70PQ - 35Q^2 \tag{27}$$

$$y = -P^2 + 2PQ + 35Q^2 \tag{28}$$

$$w^2 = 35Q^2 + P^2 \tag{29}$$

Substituting (27) and (28) in (25) and for simplicity, taking the positive sign, it is obtained that

$$x = 12PQ \tag{30}$$

In view of (23),

$$z = P^2 + 10PQ - 35Q^2 \tag{31}$$

Now, observe that (29) is satisfied by

$$Q = 2ab\ ,\ P = 35a^2 - b^2 \tag{32}$$

$$w = 35a^2 + b^2 \tag{33}$$

The substitution of (32) in (28), (30) and (31) gives

$$\left.\begin{aligned} x &= 24ab\left(35a^2 - b^2\right) \\ y &= -\left(35a^2 - b^2\right)^2 + 4ab\left(35a^2 - b^2\right) + 140a^2b^2 \\ z &= \left(35a^2 - b^2\right)^2 + 20ab\left(35a^2 - b^2\right) - 140a^2b^2 \end{aligned}\right\} \tag{34}$$

Thus, (34) and (33) represent the integer solutions to (1).

Note: 3

One may represent (29) as the system of double equations as shown in Table: 2 below:

Table 2 System of double equations

System	1	2	3	4
$w+P$	$7Q$	$7Q^2$	$5Q^2$	Q^2
$w-P$	$5Q$	5	7	35

Solving system: 1, we have

$$P = Q \tag{35}$$

$$w = 6Q \tag{36}$$

Substituting (35) in (28), (30) and (31), we have

$$y = 36Q^2\ ,\ x = 12Q^2\ ,\ z = -24Q^2 \tag{37}$$

Thus, (36) and (37) represent the integer solutions to (1).
Solving system: 2, we have

$$Q = 2s + 1\,,\ P = 14s^2 + 14s + 1 \tag{38}$$

$$w = 14s^2 + 14s + 6 \tag{39}$$

Substituting (38) in (30), (28) and (31), we have

$$\left.\begin{aligned} x &= 12\left(2s+1\right)\left(14s^2 + 14s + 1\right) \\ y &= \left(2s+1\right)\left(28s^2 + 98s + 37\right) - \left(14s^2 + 14s + 1\right)^2 \\ z &= \left(14s^2 + 14s + 1\right)^2 + \left(2s+1\right)\left(140s^2 + 70s - 25\right) \end{aligned}\right\} \tag{40}$$

Thus, (39) and (40) represent the integer solutions to (1).
Solving system: 3, we have

$$Q = 2s + 1\,,\ P = 10s^2 + 10s - 1 \tag{41}$$

$$w = 10s^2 + 10s + 6 \tag{42}$$

Substituting (41) in (30), (28) and (31), we have

$$\left.\begin{aligned} x &= 12(2s+1)(10s^2+10s-1) \\ y &= (2s+1)(20s^2+90s+33)-(10s^2+10s-1)^2 \\ z &= (10s^2+10s-1)^2+(2s+1)(100s^2+30s-45) \end{aligned}\right\} \quad (43)$$

Thus, (42) and (43) represent the integer solutions to (1).

Solving system: 4, we have

$$Q = 2s+1\,,\ P = 2s^2+2s-17 \quad (44)$$

$$w = 2s^2+2s+18 \quad (45)$$

The substitution of (44) in (28), (30) and (31) gives x, y, z and along with (45), they represent the integer solutions to (1)

It is worth mentioning that one may also introduce the transformations $x = X-2T$, $y = X-5T$ in addition to (4) and repeating the analysis presented above, different sets of integer solutions to (1) are obtained.

In this paper, an attempt has been made to determine infinitely many non-zero distinct integer solutions to the bi-quadratic equation with four unknowns given by (1). As diophantine equations are rich in variety, an attempt may be made to find integer solutions to other choices of biquadratic equations with multiple variables as well as higher order diophantine equations with multiple variables.

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