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Research Article

GENERALIZED T-HARDY ROGERS CONTRACTION THEOREMS IN CONE METRIC SPACES WITH C- DISTANCE

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ABSTRACT

A new concept of the c-distance in cone metric spaces has been introduced by Cho *et al*. [12] in 2011. Recently, Tiwari, S. K.*et al*. [30] in 2017 introduced the T-Hardy Rogers contraction under the concept of c-distance in cone metric spaces and proved uniqueness fixed point results. The purpose of this paper is to establish the generalization of T-Hardy Rogers contractive type of mapping on complete cone metric spaces. Our results generalize and extend some well known results in the literature.

INTRODUCTION

The first important fundamentals result in fixed point theory, which is also known as Banach contraction principle or Banach fixed point theorem [1].After this provital result, many authors have studied various extensions and generalizations of Banach's theorems by considering contractive mappings on several directions in the literature (see [3-11]).

In 2007, Huang and Zhang [2] generalized concept of metric space, replacing the set of real numbers by an order Banach space, and showed some fixed point theorems of different type of contractive mappings on cone metric spaces. Later, many authors generalized and studied fixed and common fixed point results in cone metric spaces for normal and non normal cone. The Hardy –Roger's contraction was introduced in the work of Hardy –Rogers [15] which is generalization of Reich contraction. Recently, Cho *et al*. [12] Wang and Guo [14] defined a concept of the c- distance in a cone metric space, which is a cone version of the w-distance of Kada *et al*.[11] and proved some fixed point theorems in ordered cone metric spaces. Then Sintunavarat *et al*. [13] generalized the Banach contraction theorem on c- distance of Cho *et al*.[12].After that, several authors studied the existence and uniqueness of the fixed point, common fixed point, coupled fixed point and common coupled fixed point problems using this distance in cone metric spaces and ordered cone metric spaces see for examples [16-27]. Quick recently, in 2017 Tiwari, S.K.*et al*.. [30] studied some fixed point theorems of T-Hardy Rogers type mappings under the concept of c- distance in complete cone metric spaces depended on another function. '

In this paper, we studied some common fixed point theorems for generalized $T -$ Hardy-Rogers contraction type mappings under the concept of c- distance in complete cone metric spaces depended on another function. Throughout this paper, we do not impose the normality condition for the cones, but the only assumption is that the cone P is solid, that is in t $P \neq \emptyset$. Our results generalize and extend the respective theorems 3.1 of the result [30].

Preliminary notes

First, we recall some standard notations and definitions in cone metric spaces with some of their properties [2].

***Definition* 2.1:** Let E be a real Banach space and P be a subset of $E$and θdenote to the zero element in E, then P is called a cone if and only if :

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i. P is a non-empty set closed and $P \neq \{\theta\}$,
ii. (ii If a, b are non-negative real numbers and $x, y \in P$,then $ax + by \in P$,
iii. $x \in P \; and - x \in P \Longrightarrow x = \theta \Leftrightarrow P \cap (-P) = \{\theta\}$.

Given a cone P ⊂ E, we define a partial ordering ≤ on E with respect to P by $x \leq y$ if and only if $y - x \in P$.We shall write $x \ll y$ if $y - x \in intP$(where int P denotes the interior of P). If $intP \neq \emptyset$, then cone P is solid. The cone P called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \; => \| x \| \leq k \| y \|.$$

The least positive number k satisfying the above is called the normal constant of P.

***Definition*: 2.2:** Let x be a non-empty set. Suppose the mapping $d: X \times X \longrightarrow E$ satisfies

i. $\theta < d(x,y) for\; all\; x, y \in X \; and \; (x,y) = \theta$if and only if $x = y$,
ii. $d(x,y) = d(y,x) for\; all\; x, y \in X$,
iii. $d(x,y) \leq d(x,z) + d(z,y) for\; all\; x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space .The concept of cone metric space is more general than that of a metric space.

Example2.3: Let $E = R^2, P = \{(x,y) \in E: x, y \geq 0\}, X = R$ and $d: X \times X \to E$ defined by $d(x,y) = (\left| x - y \right|, \alpha \left| x - y \right|)$,where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition: 2.4: Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n\geq1}$ be a sequence in X. then,

1. $\{x_n\}_{n\geq1}$ Converges to x whenever for every$c \in E$ with $\theta \ll c$, if there is a natural
2. number N such that$d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $lim_{n\to\infty} x_n =$
3. $x \; or \; x_n \to x, (n \to \infty$
4. $\{x_n\}_{n\geq1}$is said to be a Cauchy sequence if for every $c \in E$ with θ<< c,if there is a
5. natural number N such that $d(x_n, x_m) \ll c$ for all $n.m \geq N$.
6. (X, d) is called a complete cone metric space if every Cauchy sequence in X is Convergent.

***Definition 2.5*([28]):** Let(X, d) be a cone metric space, P be a solid cone and $T : X \to X$ then

a. (a) T is said to be continuous if lim $_{n\to\infty}$ $x_n = x$ implies that lim $_{n\to\infty}$ $Tx_n = Tx$ for all $\{x_n\}$ in X;
b. (b) T is said to be subsequentialy convergent, if for every sequence $\{x_n\}$ that $\{Tx_n\}$ is
c. convergent, implies $\{x_n\}$ has a convergent subsequence,
d. (c) T is said to be sequentially convergent if for every sequence $\{x_n\}$, if $\{Tx_n\}$ is convergent,
e. then $\{x_n\}$ is also convergent.
f.

***Lemma 2.6*([29])**

1. If E is a real Banach space with cone P and $a \leq \lambda a$ where $a \in P$ and $\theta \leq \lambda < 1$, then $a = \theta$
2. If $c \in intP, \theta \leq a_n$ and $a_n \to \theta$ then there a positive integer N such that $a_n \ll c$ for all $n \geq N$.

Next, we give the definition of c-distance on a cone metric space(X, d) which is generalization of w- distance of Kada *et al*. [11] with some properties.

***Definition 2.7* ([12]):** Let (X, d) be a cone metric space. A function $q: X \times X \to E$ is called a c- distance on X if the following conditions hold:

(q1). $\theta \leq q(x,y)$ for all $x, y \in X$,
(q2). $q(x,y) \leq q(x,y) + q(y,z)$ for all $x, y, z \in X$,
(q3). for each $x \in X$ and $n \geq 1$, if $q(x, y_n) \leq u$ for some$u = u_x \in P$, then $q(x,y) \leq u$

Whenever$\{y_n\}$ is a sequence in X converging to a point $y \in X$,
(q4). foe all $c \in E$with $\theta \ll c$, there exist $e \in E$ with $\theta \in e$ such that $q(z,x) \ll e$ and $q(z,y) \ll e$ imply $d(x,y) \ll c$.

***Example 2.8* ([12]):** Let $E = R$ and $P = \{x \in E: x \geq 0\}$. Let $X = [0, \infty)$and define a mapping $d: X \times X \to E$ by $d(x,y) = |x - y|$ for all $x, y \in X$. Then $(X, d)$is a cone metric space. Define by $q: X \times X \to E$ by $q(x,y) = y$ for all $x, y \in X$. Then $q$is a c-distance on$X$.

***Example 2.9*([17, 18]):** Let $E = R^2$ and $P = \{(x,y) \in E: x, y \geq 0\}$. Let$X = [0,1]$ and define a mapping $d: X \times X \to E$by $d(x,y) = (|x - y|, |x - y|)$ for all$x, y \in X$. Then (X, d) is a complete cone metric space. Define a mapping $q: X \times X \to E$ by $q(x,y) = (y, y)$ for all $x, y \in X$. Then q is a $c -$ distance.

***Example 2.10* ([26]):** Let $X = C\frac{1}{R}[0,1]$(the set of real valued functions on X which also have continuous derivatives on X),$P = \{\varphi \in E: \varphi(t) \geq 0\}$. A cone metric d on X is defined by $d(x,y)(t) \coloneqq |x - y|.\varphi(t)$ where $\emptyset \in P$ is an arbitrary function. This cone is non normal. Then $(X, d)$is a complete cone metric space. Define a mapping $q: X \times X \to E$ by $q(x,y)(t) = y.e^t$ for all $x, y \in X$. It is easy to see that q is a $c -$distance.

***Lemma 2.11*([12])**: Let (X, d) be a cone metric space and q is c- distance on X. Let $\{x_n\}$ and $\{y_n\}$ be a sequences in X and $x, y, z \in X$.Suppose that u_n is sequence in P converging to 0. Then the following conditions hold:

1. If $q(x_n, y) \leq u_n$ and $q(x_n, z) \leq u_n$, then $y = z$.
2. If $q(x_n, y_n) \leq u_n$ and $q(x_n, z) \leq u_n$, then $\{y_n\}$ converges to z.
3. If $q(x_n, x_m) \leq u_n$ for $m > n$ and $\{x_n\}$ is a Cauchy sequence in X.
4. If $q(y, x_n) \leq u_n$then $\{x_n\}$ is a Cauchy sequence in X.

***Remark 2.11*([12])**

1. $q(x,y) = q(y,x)$ does not necessarily for all $x, y \in X$.
2. $q(x,y) = \theta$ is not necessarily equivalent to $x - y$ for all $x, y \in X$.

Now, we introduce the T-Hardy –Rogers's contraction under the concept of c-distance in cone metric spaces [30].

***Definition 2.12*[30]:** Let (X, d)be a cone metric spaces and $f, T: X \to X$ be any two mappings. A mapping f is said to be T-

Hardy- Rogers contraction, if there exists a constant $k, l, m, n, r \in [0,1)$ with $k + l + m + n + r < 1$ such that

$q(Tfx, Tfy) \leq kq(Tx, Ty)$+$lq(Tx, Tfx)$+ $mq(Ty, Tfy)$ + $nq(Tx, Tfy) + rq(Ty, Tfx)$ for all $x, y \in X$.

MAIN RESULTS

Now, we give our main results in this paper.

Theorem 3.1: Let (X, d) be a complete cone metric spaces, P be a solid cone and q be a c −distance on X. In addition $T: X \to X$ be an one to one, continuous function and $R, S: X \to X$ be a pair mappings satisfies the contractive condition

$q(TFx, TGy) \leq kq(Tx, Ty)$+$lq(Tx, TFx)$+ $mq(Ty, TGy)$ + $nq(Tx, TGy)$ +$rq(Ty, TFx)$………… (3.1.1)

for all $x, y \in X$.where $k, l, m, n, r \in [0,1)$ are constants such that $k + l + m + n + r < 1$

Then F and G have an unique common fixed point $x^* \in X$. And for any $x \in X$,iterative sequence $\{F^{2n+1}x\}$ and $\{G^{2n+2}x\}$ converges to the common fixed point. If $v = Fv = Gv$. Then $q(v, v) = \theta$.

Proof: Let x_0 be an arbitrary point in X. We define the iterative sequence$\{x_{2n}\}$ and $\{x_{2n+1}\}$by

$x_{2n+1} = Fx_{2n} = F^{2n}x_0$ … (3.1.2) and
$x_{2n+2} = Gx_{2n+1} = G^{2n+1}x_0$ … (3 .1 .3).

Then, from (3.1.1), we have

$q(Tx_{2n,}Tx_{2n+1}) = q(TFx_{2n-1}, TGx_n)$
$\leq kq(Tx_{2n-1,}Tx_{2n})$ +$lq(Tx_{2n-1,}TFx_{2n-1})$+ $mq(Tx_{2n,}TGx_{2n})$ + $nq(Tx_{2n-1,}\ TGx_{2n}$)+ $rq(Tx_{2n,}\ TFx_{2n-1})$
$\leq kq(Tx_{2n-1,}Tx_{2n})$ +$lq(Tx_{n-1,}Tx_{2n})$+$mq(Tx_{2n,}Tx_{2n+1})$ + $nq(Tx_{2n-1,}\ Tx_{2n+1}$)+ $rq(Tx_{2n,}\ Tx_{2n})$
$q(Tx_{2n,}Tx_{2n+1}) \leq (k + l + n)q(Tx_{2n-1,}Tx_{2n})$+$(m + n)q(Tx_{2n,}Tx_{2n+1})$
=>$([1 - (m + n)]q(Tx_{2n,}Tx_{2n+1}) \leq (k + l + n)q(Tx_{2n-1,}Tx_{2n})$
=> $q(Tx_{2n,}Tx_{2n+1}) \leq \frac{k+l+n}{1-(m+n)} q(Tx_{2n-1,}Tx_{2n})$
$\leq hq(Tx_{2n-1,}Tx_{2n}$ … (3.1.2)

Where $\frac{k+l+n}{1-(m+n)} = h < 1$.

Let $m > n \geq 1$. we have

$q(Tx_{n,}\ Tx_m) \leq q(Tx_{n,}Tx_{n+1})$ + $q(Tx_{n+1,}Tx_{n+2})$ +……….+$q(Tx_{n-1,}Tx_n)$
$\leq (h^n + h^{n+1} + \cdots \ldots \ldots \ldots \ldots \ldots + h^{n-1})q(Tx_{0,}Tx_1)$
$\leq \frac{h^n}{1-h} q(Tx_{0,}Tx_1) \to \infty, h \to \infty.$

Thus, Lemma 2.11(3), which implies that, $\{TFx_{2n}\}$ is a Cauchy sequence in X. Since X is complete cone metric space, then there exist $v \in X$ such that

$Tx_{2n} \to u$ as $n \to \infty$ ….(3.1.3)

Since T is subsequently convergent, $\{x_{2n}\}$ has a convergent subsequence. So, there are $x^* \in X$ and $\{x_{2ni}\}$ such that
$x_{2ni} \to x^*$ as $i \to \infty$. … (3.1.7)

Since T is con tenuous, then by (3.1.6), we obtain
$Tx_{2i} = Tx^*$…. (3.1.8)

Now from (3.1.6) and (3.1.8), we conclude that
$Tx^* = u$… (3.1.9)

By definition [2.7] (q3), we have

$q(Tx_{2n,}\ Tx^*) \leq \frac{h^{2n}}{1-h} q(Tx_{0,}Tx_1)$… (3.1.7)

On the other hand and using (3.1.5), we have

$q(Tx_{n,}\ TFx^*) \leq q(TFx_{2n-1,}TFx^*)$
$\leq kq(Tx_{2n-1}, Tx^*)$
$\leq k \frac{h^{2n-1}}{1-h} q(Tx_{0,}Tx_1)$
$= \frac{h^{2n}}{1-h} q(Tx_{0,}Tx_1)$… (3.1.8)

By lemma 2.11(1), from (3.1.7) and (3.1.8), we have
$Tx^* = TFx^*$… (3.1.12)

Since T is one to one, then $x^* = Fx^*$. Thus $x^*$is a fixed point of F. Similarly, we can prove that x^* is a fixed point of G. Therefore, x^* is common fixed point of F and G.
Moreover, suppose that, $v = Rv = Sv$, and then we have

$q(Tv, Tv) = q(TFv, TGv)$
$\leq kq(Tv, Tv) + lq(Tv, TFv)$+$mq(Tv, TGv)$+$nq(Tv, TGv)$
+$rq(Tv, TFv)$
$= (k + l + m + n + r) q(Tv, Tv)$

Since $k + l + m + n + r < 1$, lemma 2.6 (1), shows that $q(Tx^*, Ty^*) = \theta$.

Finally suppose that, if y^* is another common fixed point of F and G. Then we have

$q(Tx^*, Ty^*) = q(TFx^*, TGy^*$
$\leq kq(Tx^*, Ty^*) + lq(Tx^*, TFx^*)$+$mq(Ty^*, TGy^*)$

+ $nq(Tx^*, TGy^*)$+ $lq(Ty^*, TFx^*)$+
$= kq(x^*, y^*) + lq(x^*, x^*) + mq(y^*, y^*)$+$nq(x^*, y^*)$+$rq(y^*, x^*)$
$= (k + n + r)q(Tx^*, Ty^*)$.
$\leq (k + l + m + n + r)q(Tx^*, Ty^*)$.

'Since $k + l + m + n + r < 1$, lemma2.6 (1), shows that $q(Tx^*, Ty^*) = \theta$.Also we have $q(Tx^*, Tx^*) = \theta$.Thus, Lemma 2.11(1), $Tx^* = Ty^*$. Since T is one to one, then $x^* = y^*$. So, x^* is the unique common fixed point of F and G.

If we take $T = 1$ and $F = G = f$ in the above theorem, we get the following results of corollary [3.3] of [30].

***Corollary* 3.2:** Let (X, d) be a complete cone metric spaces, P be a solid cone and q be a c −distance on X. Let $f: X \to X$ be a mappings satisfies the contractive condition

$q(fx, fy) \leq kq(x, y)$+$lq(x, fx)$+ $mq(y, fy) + nq(x, fy)$ + $rq(y, fx)$

for all $x, y \in X$.where $k, l, m, n, r \in [0,1)$ are constants such that $k + l + m + n + r < 1$
Then f has an unique fixed point $x^* \in X$. And for any $x \in X$,iterative sequence $\{fx_n\}$ converges to the fixed point. If $u = fu$. Then $q(u, u) = \theta$.

CONCLUSION

In this attempt, we prove unique common fixed point results in cone metric spaces with corollaries. These results generalizes and improves the recent results of Tiwari,S .K. *et al*. [30] in the sense that employing c-distances and in contractive conditions, which extends the further scope of our results.

Conflict of Interests

The authors declare that there is no conflict of interests.

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