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CODEN: IJRSFP (USA)

International Journal of Recent Scientific Research
Vol. 10, Issue, 05(E), pp. 32451-32453, May, 2019

ISSN:0976-3031

International Journal of Recent Scientific Research

DOI: 10.24327/IJRSR

Research Article

MUTUALLY UNBIASED UNEXTENDIBLE MAXIMALLYENTANGLED BASES IN $C^2 \otimes C^3$

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DOI: http://dx.doi.org/10.24327/ijrsr.2019.1005.3473

ARTICLE INFO

Article History:
Received 13th February, 2019
Received in revised form 11th March, 2019
Accepted 8th April, 2019
Published online 28th May, 2019

Key Words:

maximally entangled states； mutually unbiased bases； unextendible maximally entangled bases； Gell-Mann matrices

ABSTRACT

Firstly, this paper uses the Gell-Mann matrices to construct 4-member unextendible maximally entangled basis in $C^2 \otimes C^3$, whose construction is different from those in literatures, then adds two product states to make it complete. Finally, by changing the basis of C^3, this paper constructs a pair of mutually unbiased unextendible maximally entangled bases $C^2 \otimes C^3$.

INTRODUCTION

Quantum entanglement is a basic phenomenon of quantum physics, which plays a very important role in quantum information processing, there has been extensive research in recent years [1-3]. A state $|\psi\rangle$ is said to be a $C^d \otimes C^{d'}$ maximally entangled state [4] if and only if for an arbitrary given orthonormal complete basis $\{|i_A\rangle\}$ of the subsystem A, there exist an orthonormal basis $\{|i_B\rangle\}$ of the subsystem B such that $|\psi\rangle$ can be written as $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i_A\rangle \otimes |i_B\rangle$.

The unextendible maximally entangled bases (UMEB)was studied in arbitrary bipartite spaces $C^d \otimes C^{d'}$ $(\frac{d'}{2} < d < d')$ in [5], and two mutually unbiased complete UMEBs were constructed. Multi pairs of mutually unbiased UMEB in $C^2 \otimes C^3$ were studied [6-9]. The method of systematically constructing mutually unbiased unextendible maximally entangled bases(MUUMEBs) was given in [10]. All the above MUUMEBs was constructed with the help of Bell basis and Pauli matrix, while this paper will construct MUUMEBs using 3×3 Gell-Mann matrices.

UMEB in $C^2 \otimes C^3$

Definition 1. [11] Two orthogonal bases $\mathcal{B}_1 = \{|\varphi_i\rangle\}_{i=1}^d$ and $\mathcal{B}_2 = \{|\psi_j\rangle\}_{j=1}^d$ of C^d are called mutually unbiased if

$$|\langle \varphi_i | \psi_j \rangle| = \frac{1}{\sqrt{d}} \ (1 \le i, j \le d)$$

A set of orthonormal bases $\{\mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_m\}$ in C^d is said to be a set of mutually unbiased bases (MUBs) if every pair of $\mathcal{B}_i$ and $\mathcal{B}_j$ $(1 \le i \ne j \le m)$ in the set is mutually unbiased.

Definition2. [5] A set of states $\{|\varphi_i\rangle \in C^d \otimes C^{d'} (d \le d') : i = 1, 2, \cdots, n, n < dd'\}$ is called an n -number UMEB if it satisfies the following conditions:

1. $|\varphi_i\rangle$, $i = 1, 2, \cdots, n$ are all maximally entangled;
2. $\langle \varphi_i | \varphi_j \rangle = \delta_{ij}$;

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This work is supposed by Natural Science Foundation of China under number 11761073

3. If $\langle \varphi_i | \psi \rangle = 0$ for all $i = 1, 2, \cdots, n$, then $|\psi\rangle$ cannot be maximally entangled.

Gell-Mann matrices a basic representation of infinitesimal generators SU(3) groups, which is a set of linearly independent 3×3 untracked Hermitian matrices, it was used to study strong interactions in particle physics. Specifically, there are 8 Gell-Mann matrices as follows:

$$H_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$H_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$H_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad H_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$H_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad H_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

First, we construct UMEB in $C^2 \otimes C^3$ using Gell-Mann matrices. Let $\{|0\rangle, |1\rangle\}, \{|0'\rangle, |1'\rangle, |2'\rangle\}$ are the orthonormal bases in C^2 and C^3 respectively, and suppose that $|\varphi_0\rangle = \frac{1}{\sqrt{2}}(|00'\rangle + |11'\rangle)$. Consider the following four maximally entangled states in $C^2 \otimes C^3$:

$$|\varphi_i\rangle = (I \otimes H_i)|\varphi_0\rangle \;\; (i = 0, 1, 2, 3) \qquad (1)$$

where I_2 denotes the 2×2 identity matrix, H_0 stands for the 3×3 identity matrix, H_1, H_2, H_3 are the first three Gell-Mann matrices.

Theorem 1 The four states in equation (1) constitute a 4-number UMEB in $C^2 \otimes C^3$.

Proof. Obviously, the four states in $C^2 \otimes C^3$ are maximally entangled and mutually orthonormal. We should prove that if there exists state $|\psi\rangle$ such that $\langle \varphi_i | \psi \rangle = 0 \; (i = 0, 1, 2, 3)$, then $|\psi\rangle$ must not be maximally entangled.

If $|\psi\rangle$ is an entangled states, $|\psi\rangle$ must have the following Schmidt decomposition:

$$|\psi\rangle = (U \otimes V)(\lambda_0 |00'\rangle + \lambda_1 |11'\rangle)$$

where $\lambda_0 > 0, \lambda_1 > 0, \lambda_0^2 + \lambda_1^2 = 1$; U and V are the following unitary operators,

$$U = (u_{ij})_{2\times 2} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix},$$

$$V = (v_{ij})_{3\times 3} = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}.$$

From $\langle \varphi_0 | \psi \rangle = 0$ we get that

$$\begin{aligned} 0 &= \frac{1}{\sqrt{2}}(\langle 00'| + \langle 11'|)(U \otimes V)(\lambda_0 |00'\rangle + \lambda_1 |11'\rangle) \\ &= \frac{\lambda_0}{\sqrt{2}}\langle 0|U|0\rangle\langle 0'|VH_0|0'\rangle + \frac{\lambda_1}{\sqrt{2}}\langle 0|U|1\rangle\langle 0'|VH_0|1'\rangle + \frac{\lambda_0}{\sqrt{2}}\langle 1|U|0\rangle\langle 1'|VH_0|0'\rangle + \frac{\lambda_1}{\sqrt{2}}\langle 1|U|1\rangle\langle 1'|VH_0|1'\rangle \end{aligned} \qquad (2)$$

Similarly, we can derive the following equations from $\langle \varphi_i | \psi \rangle = 0 \; (i = 1, 2, 3)$ respectively,

$$\begin{aligned} 0 &= \frac{1}{\sqrt{2}}(\langle 00'| + \langle 11'|)(I \otimes H_1)(U \otimes V)(\lambda_0 |00'\rangle + \lambda_1 |11'\rangle) \\ &= \frac{\lambda_0}{\sqrt{2}}\langle 0|U|0\rangle\langle 0'|VH_1|0'\rangle + \frac{\lambda_1}{\sqrt{2}}\langle 0|U|1\rangle\langle 0'|VH_1|1'\rangle + \frac{\lambda_0}{\sqrt{2}}\langle 1|U|0\rangle\langle 1'|VH_1|0'\rangle + \frac{\lambda_1}{\sqrt{2}}\langle 1|U|1\rangle\langle 1'|VH_1|1'\rangle \end{aligned} \qquad (3)$$

$$\begin{aligned} 0 &= \frac{1}{\sqrt{2}}(\langle 00'| + \langle 11'|)(I \otimes H_2)(U \otimes V)(\lambda_0 |00'\rangle + \lambda_1 |11'\rangle) \\ &= \frac{\lambda_0}{\sqrt{2}}\langle 0|U|0\rangle\langle 0'|VH_2|0'\rangle + \frac{\lambda_1}{\sqrt{2}}\langle 0|U|1\rangle\langle 0'|VH_2|1'\rangle + \frac{\lambda_0}{\sqrt{2}}\langle 1|U|0\rangle\langle 1'|VH_2|0'\rangle + \frac{\lambda_1}{\sqrt{2}}\langle 1|U|1\rangle\langle 1'|VH_2|1'\rangle \end{aligned} \qquad (4)$$

$$\begin{aligned} 0 &= \frac{1}{\sqrt{2}}(\langle 00'| + \langle 11'|)(I \otimes H_3)(U \otimes V)(\lambda_0 |00'\rangle + \lambda_1 |11'\rangle) \\ &= \frac{\lambda_0}{\sqrt{2}}\langle 0|U|0\rangle\langle 0'|VH_3|0'\rangle + \frac{\lambda_1}{\sqrt{2}}\langle 0|U|1\rangle\langle 0'|VH_3|1'\rangle + \frac{\lambda_0}{\sqrt{2}}\langle 1|U|0\rangle\langle 1'|VH_3|0'\rangle + \frac{\lambda_1}{\sqrt{2}}\langle 1|U|1\rangle\langle 1'|VH_3|1'\rangle \end{aligned} \qquad (5)$$

Simplify the equations (2), (3), (4) and (5) as follows,

$$\begin{cases} \lambda_0 u_{11} v_{11} + \lambda_1 u_{12} v_{12} + \lambda_0 u_{21} v_{21} + \lambda_1 u_{22} v_{22} = 0 \\ \lambda_0 u_{21} v_{11} + \lambda_1 u_{22} v_{12} + \lambda_0 u_{11} v_{21} + \lambda_1 u_{12} v_{22} = 0 \\ \lambda_0 u_{21} v_{11} + \lambda_1 u_{22} v_{12} - \lambda_0 u_{11} v_{21} - \lambda_1 u_{12} v_{22} = 0 \\ \lambda_0 u_{11} v_{11} + \lambda_1 u_{12} v_{12} - \lambda_0 u_{21} v_{21} - \lambda_1 u_{22} v_{22} = 0 \end{cases} \qquad (6)$$

Obviously, the equations (6) can be expressed as:

$$\begin{pmatrix} u_{11} & u_{12} & u_{21} & u_{22} \\ u_{21} & u_{22} & u_{11} & u_{12} \\ u_{21} & u_{22} & -u_{11} & -u_{12} \\ u_{11} & u_{12} & -u_{21} & -u_{22} \end{pmatrix} \begin{pmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \lambda_0 & \\ & & & \lambda_1 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \\ v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad (7)$$

Suppose $A = \begin{pmatrix} U & MU \\ MU & -MU \end{pmatrix} \begin{pmatrix} W & \\ & W \end{pmatrix}$,

where $W = \begin{pmatrix} \lambda_0 & \\ & \lambda_1 \end{pmatrix}, M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, v = \begin{pmatrix} v_{11} \\ v_{12} \\ v_{21} \\ v_{22} \end{pmatrix}$,

then equation (7) can be expressed as

$$Av = 0. \qquad (8)$$

Since

$$\det(A) = \begin{vmatrix} U & MU \\ MU & -U \end{vmatrix} \begin{vmatrix} W & \\ & W \end{vmatrix} = \begin{vmatrix} U & MU \\ 0 & -2U \end{vmatrix} |W|^2 = 4\lambda_0^2 \lambda_1^2 (\det U)^2 \neq 0,$$

then equation (8) only has zero solution $v = 0$, i.e., $v_{11} = v_{12} = v_{21} = v_{22} = 0$, Thus,

$$\det(V)=\begin{vmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{vmatrix}=\begin{vmatrix} 0 & 0 & v_{13} \\ 0 & 0 & v_{23} \\ v_{31} & v_{32} & v_{33} \end{vmatrix}=0$$

, that is to say, V is not an unitary matrix, which contradicts the hypothesis. Therefore, $|\psi\rangle$ must not be an entangled state.

To sum up, the four states in (1) constitute a 4-number UMEB in $C^2 \otimes C^3$.

MUUMEBs in $C^2 \otimes C^3$

Add the following two product states to equation (1), we can get a complete UMEB in $C^2 \otimes C^3$:

$$\begin{cases} |\varphi_4\rangle = \frac{1}{2}|02'\rangle + \frac{\sqrt{3}}{2}|12'\rangle \\ |\varphi_5\rangle = \frac{\sqrt{3}}{2}|02'\rangle + \frac{1}{2}|12'\rangle \end{cases} \quad (9)$$

Let $\{|x'\rangle, |y'\rangle, |z'\rangle\}$ be another orthonormal basis in C^3 and

$$\begin{cases} |x'\rangle = \frac{1}{\sqrt{3}}(|0'\rangle + \frac{1+\sqrt{3}i}{2}|1'\rangle + i|2'\rangle) \\ |y'\rangle = \frac{1}{\sqrt{3}}(\frac{-\sqrt{3}+i}{2}|0'\rangle + i|1'\rangle + |2'\rangle) \\ |z'\rangle = \frac{1}{\sqrt{3}}(i|0'\rangle - i|1'\rangle + \frac{1+\sqrt{3}i}{2}|2'\rangle) \end{cases}$$

where $i=\sqrt{-1}$.

Using the method in Section 2, we can get another UMEB in $C^2 \otimes C^3$,

$$\begin{cases} |\psi_j\rangle = \frac{1}{\sqrt{2}}(I\otimes H_j)(|0x'\rangle + |1y'\rangle),\ j=0,1,2,3, \\ |\psi_4\rangle = \frac{1}{\sqrt{2}}(\frac{1+\sqrt{3}i}{2}|0z'\rangle + \frac{\sqrt{3}-i}{2}|1z'\rangle), \\ |\psi_5\rangle = \frac{1}{\sqrt{2}}(\frac{\sqrt{3}-i}{2}|0z'\rangle + \frac{1+\sqrt{3}i}{2}|1z'\rangle), \end{cases} \quad (10)$$

Using Definition 1, it is easily to verify that the above two UMEBs(9)and(10)are mutually unbiased.

CONCLUSION

In this paper, we extended the construction of UMEB in [5], and give a new construction of UMEB in $C^2 \otimes C^3$. This method makes it possible to construct UMEB of arbitrary bipartite system with higher dimensions by using Gell-Mann matrices.

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