Available Online at http://www.recentscientific.com
CODEN: IJRSFP (USA)
International Journal of Recent Scientific Research
Vol. 10, Issue, 05(G), pp. 32606-32612, May, 2019
ISSN: 0976-3031
International Journal of Recent Scientific Research
DOI: 10.24327/IJRSR

Research Article

STUDY OF SOME ITERATIVE METHODS FOR SOLVING NON-LINEAR EQUATIONS IN ONE VARIABLE

Mohammadi Khan Mohammadi

Department of Mathematics (UIS), Chandigarh University, Gharuan, Mohali, Punjab, 140413, India.

DOI: http://dx.doi.org/10.24327/ijrsr.2019.1005.3507

ARTICLE INFO

Article History:
Received 15th February, 2019
Received in revised form 7th March, 2019
Accepted 13th April, 2019
Published online 28th May, 2019

Key Words:

bracketing methods, Open end methods, rate of convergence, non-linear equation.

ABSTRACT

This paper is based on the relative study of several recognized iterative methods named Bisection, Regula-Falsi (R-F) or false position, Secant, Newton-Raphson (N-R) and Muller methods. The rate of convergence of every method will be analyzed after solving numerical problem by implementing each method independently. We solve non-linear equations in one variable by using the above iterative methods in MATLAB version R2010asoftware and find the value of a single real root.

INTRODUCTION

In mathematics we deals with many polynomial equations of the form $f(x) = a_0x^r + a_1x^{r-1} + \cdots + a_{r-1} + a_r$, where a's are constants, $a_0 \neq 0$. If $f(x)$ constant some other functions such that as trigonometric, logarithmic, exponential etc. then $f(x) = 0$ is a transcendental equation. In scientific and engineering work, a frequently occurring problem is to find the root of equation of the form$f(x) = 0$ [3]. If f be a continuous function. Any number γ for $f(\gamma) = 0$ is a root of equation $f(x) = 0$.where, root of $f(x)$isγ. A root of γ is called of multiplicity q, if $f(x) = (x - \gamma)^q g(x)$, $g(x)$ is bounded at γ and $g(\gamma) \neq 0$. If $q = 1$, then $\gamma$is said to be simple zero and if $q > 1$, then γ is called a multiple zero [4].

Noor and Ahmad (2006) gave a predictor correction type iterative method to solve $f(x) = 0$ by using a method consists of Regula-Falsi (R-F) and Newton-Raphson (N-R)method. On performing the numerical experiment, the new predictor correction method was for better than the method known at the time.Noor et al. (2006) proposed that two-step techniques are more useful than one step techniques including the Newton method. Naghipoor et al. (2008) gave a developed (R-F) method by using the classical (R-F) method and showed that the suggested method was more efficient as compared to the classical (R-F) method. Shaw and Mukhopadhyay (2015) presented in their paper an improved (R-F) method as predictor-corrector form. The method converges very fast than the previous (R-F) method. Unlike the improved (R-F) method discussed in Naghipoor (2008) paper, Shaw and Mukhopadhyay (2015) selected the value of only one parameter (k) from outside. So, the CPU time and the procedures for the implementation of this algorithm are very less.

Parida and Gupta (2006) suggested a combined method of common (R-F) and Newton-like to determine the non-linear equations. This new method is examined on various examples and results presented that the suggested method is beneficial as compare to some present methods applied to solve the same problems. Li and Chen (2006) proposed a method to determine the non-linear equations containing of the classical (R-F) method and some parameters of exponential (R-F) method with higher-order convergence for solving the single root of $f(x) = 0$. The sequence of both diameters and iterative pointes are quadratically convergent in this beneficial method

Li and Chen (2007) in their paper suggested a combined method of classical (R-F) and exponential iterative methods with high order convergence for determining the single root of nonlinear equations. The proposed method has good asymptotic quadratic convergence.

Alojz (2012) suggested a bracketing algorithm for the non-linear equation with the iterative zero findings. The well-

Corresponding author:* **Mohammadi Khan Mohammadi
Department of Mathematics (UIS), Chandigarh University, Gharuan, Mohali, Punjab, 140413, India.

specified bracketing methods can be relocating with the recommended algorithm in this paper. This procedure is based on (R-F) and bisection methods with the second order polynomial interpretation techniques. This method alongside with increment speed of convergent confirms global convergence. Alojz (2013) proposed a method based on Muller's algorithm which assures universal convergence, along with classic Muller, bracketing is introduced for solving non-linear equation .the advanced algorithm is convergent, subset and firm it's the more significant merit alongside with global convergence is its easiness of algorithm which is not compose of complex combinations of methods.We discuss some popular iterative methods to find out the solution of $f(x) = 0$ in one variable.

Bisection (or Bolzano) Method

It is also called binary chopping or half-interval method. For resolving $f(x) = 0$, the bisection method is one of the easy and most valid iterative procedures. It is based on intermediate value property, i.e. whether $f(x)$ is real and continuous in $(a, b)$and $f(a), f(b)$ are opposed signs, then ∃ at least one root in (a, b) such that

$$x_0 = \frac{a+b}{2}$$

Now the following three cases arise

I. If $f(x_0) = 0$, then x_0 is root of $f(x)$.

II. If $f(x_0) > 0$, then root of $f(x_0)$ will lie between a and x_0 that is,
$$x_1 = \frac{a+x_0}{2}$$

III. If $f(x_0) < 0$, then root will lie between x_0 and b, that is
$$x_1 = \frac{x_0+b}{2}$$

Suppose $f(x_0) > 0$, then the new interval is $[a, x_0]$ with $length = |x_0 - a|$, but length of previous interval is $|b - a|$, that is

$$|x_0 - a| = \left|\frac{a+b}{2} - a\right| = \left|\frac{b-a}{2}\right|$$

Again apply intermediate value property, get a new interval with length as half of $[a, x_0]$. We repeat above procedure until the interval which contains the root is very small, sayε. As interval length becomes half after every step. Let at nth step, the interval is $[a_n, b_n]$ with length $\left|\frac{b-a}{2^n}\right|$, we have

$$\left|\frac{b-a}{2^n}\right| \le \varepsilon$$

This gives

$$n \ge \frac{log_e^2 \left|\frac{b-a}{2^n}\right|}{log_e^2}$$

Therefore, if we know the value of $|b - a|$, and e then number of iterations can be found by this formula.

Convergence of Bisection Method

In this method, the original interval is broken into half interval in each of the iterations if we use the midpoints of the successive interval to be the approximation of the root, the one half of the current interval is the upper bound to the error.

$$e_{j+1} = 0.5e_j$$

Implies that

$$\frac{e_{j+1}}{e_j} = \frac{1}{2} \qquad (1)$$

Where e_j and e_{j+1} are the errors in the j^{th} and$(e_{j+1})^{th}$iteration. Comparing equation (1) with

$$\lim_{j\to\infty} \left|\frac{e_{j+1}}{e_j}\right| \le M$$

Then, we have $\beta = 1$ and $M = 1/2$ or 0.5

So, this method is 1st order convergence or linear order convergent

Regula-Falsi Method or Method of False Position

This is the oldest method if we want to find out the root of $f(x) = 0$ and it is approximately similar to the bisection method. It is also called a method of chords or a method of linear interpolation. In this case, we choose two-point, i.e. $f(x_0)$and $f(x_1)$are of opposing signs. As $y = f(x)$ passes x-axis among two points, therefore a zero must lie among these two points subsequently, $f(x_0) \cdot f(x_1) < 0$. Now we joining the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ by the straight line and suppose the point where this line intersects the x-axis is the next estimate to the root, we assume that the line crosses the x-axis at x_2. If $f(x_1)$ and $f(x_2)$ are opposed signs, thus x_1 is replaced by x_2 and to find the crosses point, we joining $f(x_2)$ and $f(x_0)$ by a straight line. If $f(x_1)$ and $f(x_2)$ are of the same opposite signs, thus x_0 is replaced by x_2 and the iterative procedure is repeated. In both cases, the previous search interval is bigger than the new search interval and ultimately this will converge to a root. From the slope of the line, we get

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = tang\beta = \frac{f(x_1) - 0}{x_1 - x_2}$$

$$\Rightarrow x_2 = \frac{x_1[f(x_1) - f(x_0)] - f(x_1)[x_1 - x_0]}{f(x_1) - f(x_0)}$$

$$\Rightarrow x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \qquad (2)$$

Which is gives an approximation to the root.

This procedure reiterated till the root is established to the desired precision.

In general, for the $(j+1)^{th}$guess to the root is replacing $x_0$by x_{j-1}, x_1 by x_j and x_2 by x_{j+1}so, equation (2) becomes

$$x_{j+1} = \frac{x_{j-1} f(x_j) - x_j f(x_{j-1})}{f(x_j) - f(x_{j-1})} \qquad (3)$$

Relation (3) is general formula for Method of False position.

Order (rate) of convergence for Method of False position

Let any number γ for $f(\gamma) = 0$ is precise root of equation $f(x) = 0$, and x_j deferent from γ by e_j is small quantity. Similarly $x_{j-1}$and x_{j+1} are deferent from γ by which e_{j-1} and e_{j+1} are also small quantity. Now we have

$$x_{j-1} = e_{j-1} + \gamma, \qquad x_j = e_j + \gamma, \qquad x_{j+1} = e_{j+1} + \gamma \tag{4}$$

From (3) and (4), we get

$$e_{j+1} = \frac{e_{j-1} f(e_j + \gamma) - e_j f(e_{j-1} + \gamma)}{f(e_j + \gamma) - f(e_{j-1} + \gamma)} \tag{5}$$

Now by apply Taylor's numerator, expanding $f(e_j + \gamma) and\ f(e_{j-1} + \gamma)$ of (5) is

$$e_{j-1} f(e_j + \gamma) - e_j f(e_{j-1} + \gamma) = e_{j-1}\left[f(\gamma) + \frac{e_j}{1!} f'(\gamma) + \frac{e_j^2}{2!} f''(\gamma) + \cdots\right] - e_j\left[f(\gamma) + \frac{e_{j-1}}{1!} f'(\gamma) + \frac{(e_{j-1})^2}{2!} f''(\gamma) + \cdots\right]$$

Since γ is the zero of $f(x) = 0$. As $f(\gamma) = 0$ and ignoring higher degree terms, we have

$$= \frac{e_{j-1} \cdot e_j^2}{2!} f''(\gamma) - \frac{e_j (e_{j-1})^2}{2!} f''(\gamma)$$

So e_j is small, we neglectinge_j^2, $(e_{j-1})^2$ and higher degree terms, we get

$$e_{j-1} f(e_j + \gamma) - e_j f(e_{j-1} + \gamma) = \frac{e_{j-1} \cdot e_j (e_j - e_{j-1})}{2!} \cdot f''(\gamma) \tag{6}$$

Again the denominator of (5) is

$$f(e_j + \gamma) - f(e_{j-1} + \gamma) = \left[f(\gamma) + \frac{e_j}{1!} f'(\gamma) + \frac{e_j^2}{2!} f''(\gamma) + \cdots\right] - \left[f(\gamma) + \frac{e_{j-1}}{1!} f'(\gamma) + \frac{(e_{j-1})^2}{2!} f''(\gamma) + \cdots\right]$$

So, we neglectinge_j^2, $(e_{j-1})^2$and higher order terms, we have

$$f(e_j + \gamma) - f(e_{j-1} + \gamma) = (e_j - e_{j-1}) f'(\gamma) \tag{7}$$

Using (6) and (7) equation (5) becomes

$$e_{j+1} = \frac{1/2! \cdot e_{j-1} \cdot e_j (e_j - e_{j-1})}{(e_j - e_{j-1})} \cdot \frac{f''(\gamma)}{f'(\gamma)}$$

$$e_{j+1} = e_{j-1} e_j \cdot k \tag{8}$$

Where $\frac{f''(\gamma)}{2f'(\gamma)} = k$ is a finite constant.

Let β be the rate (order) of convergence, then we have

$$e_j \le e_{j-1}^{\beta} \cdot k'$$

or taking

$$e_j = e_{j-1}^{\beta} \cdot k' \tag{9}$$

Eliminating e_{j-1} from (8) and (9), we have

$$e_{j+1} = \left(\frac{e_j}{k}\right)^{1/\beta} \cdot e_j k = e_j^{1+1/\beta} \cdot \frac{k}{(k')^{1/\beta}} \tag{10}$$

Also

$$e_{j+1} = e_j^{\beta} k' \tag{11}$$

The value of e_{j+1} equation from (10) and (11), we have

$$e_j^{1+1/\beta} \cdot \frac{k}{(k')^{1/\beta}} = e_j^{\beta} \cdot k' \tag{12}$$

Now choosing $k\ and\ k'$ so that $k' = \frac{k}{(k')^{1/\beta}}$

$$k = k'(k')^{1/\beta} = (k')^{1+1/\beta} \tag{13}$$

That is equation (13) becomes

$$e_j^{1+1/\beta} = e_j^{\beta}$$

$$\Rightarrow \quad 1 + 1/\beta = \beta \quad or \quad \beta^2 - \beta - 1 = 0$$

$$\Rightarrow \quad \beta = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Choosing +ve sign, we have

$$\beta = \frac{1 + \sqrt{5}}{2} = \frac{3.236}{2} = 1.618$$

Therefore it is rate (order) of convergence of Method of False position.

Newton-Raphson (N-R) method

When the derivative of $f(x)$, can be easily found, by the process of Newton-Raphson method the correct zero of the equation $f(x) = 0$ can be computed. Let x_j be an estimate to the zero of $f(x) = 0$. SupposeΔx be an enhancement in x, i.e. $x_j + \Delta x$ is a correct zero. Such that

$$f(x_{j+1} + \Delta x) \equiv 0$$

Expanding $f(x_j + \Delta x)$ by Taylor's series the point x_j, then we have

$$f(x_j) + \frac{\Delta x}{1!} f'(x_j) + \frac{(\Delta x)^2}{2!} f''(x_j) + \cdots = 0$$

Because(Δx) is a very small quantity, then ignoring$(\Delta x)^2$ and higher powers, so we get

$$f(x_j) + \Delta x f'(x_j) = 0$$

$$\Rightarrow \quad \Delta x = -\frac{f(x_j)}{f'(x_j)}$$

Hence, we obtain the iteration method, we have

$$x_{j+1} = x_j + \Delta x = x_j - \frac{f(x_j)}{f'(x_j)}$$

$$x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)}, \quad (j = 0,1,2,\dots) \qquad (14)$$

Equation (14) is called Newton- Raphson formula.

Order of convergence of the Newton-Raphson method

Let any number γ for $f(\gamma) = 0$ is precise zero of equation$f(x) = 0$, and e_j small quantity by which x_j deferent fromγ, similerly, e_{j+1} is a quantity by which x_{j+1} wary for γ , then we have

$$x_j = \gamma + e_j, \qquad x_{j+1} = \gamma + e_{j+1} \qquad (15)$$

From (15) equation (14) become

$$e_{j+1} = e_j - \frac{f(\gamma + e_j)}{f'\left(\gamma + e_{j+1}\right)} \qquad (16)$$

Expanding $f(\gamma + e_j) and\ f'\left(\gamma + e_{j+1}\right)$ by Taylor's series equation (4) is

$$e_{j+1} = e_j - \frac{f(\gamma) + \frac{e_j}{1!} f'(\gamma) + \frac{e_j^2}{2!} f''(\gamma) + \cdots}{f'(\gamma) + \frac{e_j}{1!} f''(\gamma) + \cdots}$$

Since γ is the zero of $f(x) = 0$, as $f(\gamma) = 0$, we have

$$e_{j+1} = e_j - \frac{\frac{e_j}{1!} f'(\gamma) + \frac{e_j^2}{2!} f''(\gamma) + \cdots}{f'(\gamma) + \frac{e_j}{1!} f''(\gamma) + \cdots}$$

So e_j is small, therefore neglecting e_j higher order terms, we get

$$e_{j+1} = e_j - \frac{[e_j f'(\gamma) + \frac{e_j^2}{2} f''(\gamma)]}{[f'(\gamma) + e_j f''(\gamma)]}$$

$$e_{j+1} = \frac{e_j^2 f''(\gamma)}{2f'(\xi)[1 + e_j \frac{f''(\gamma)}{f'(\gamma)}]} = \frac{e_j^2 f''(\gamma)}{2f'(\gamma)} \left[1 + e_j \frac{f''(\gamma)}{f'(\gamma)}\right]^{-1}$$

Using binomial expansion, we have

$$e_{j+1} = \frac{e_j^2 f''(\gamma)}{2f'(\gamma)} \left[1 - e_j \frac{f''(\gamma)}{f'(\gamma)} + \cdots\right]$$

Ignoring the higher order term, we have

$$e_{j+1} = \frac{e_j^2 f''(\gamma)}{2f'(\gamma)}$$

Now we put $\frac{f''(\gamma)}{2f'(\gamma)} = k$, where k finite constant, we have

$$e_{j+1} = k e_j^2$$

This implies

$$\frac{e_{j+1}}{e_j^2} = k$$

Comparing with

$$\lim_{j\to\infty} [\frac{e_{j+1}}{e_j^{\beta}}] \le k$$

Since the index of e_j is 2, then the rateof convergence of (N-R) method is 2. So this is a quadratic convergent.

Secant Method

Newton-Raphson method is very powerful and it has big weakness, but the evaluation of derivative involved occasionally is difficult, thus recommended the idea of changing the derived $f'(x_j)$ in Newton-Raphson formula given below

$$x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)} \qquad (17)$$

In this method derived can be estimated by a backward finite divided deference

$$f'(x_j) \cong \frac{f(x_j) - f(x_{j-1})}{x_j - x_{j-1}} \qquad (18)$$

From (17) and (18), we get

$$x_{j+1} = x_j - \frac{x_j - x_{j-1}}{f(x_j) - f(x_{j-1})} f(x_j), \quad j \ge 1, \qquad (19)$$

Equation (19) is called secant method formula. This method almost same as method of False position, but in this method it does not require the condition $f(x_0)f(x_1) < 0$.

Order of convergence of secant method

Let any number γ for $f(\gamma) = 0$ is precise root of equation$f(x) = 0$, and e_j the error in the guess of x_j , then we have

$$x_{j+1} = \gamma + e_{j+1}, \qquad x_j = \gamma + e_j \ , \qquad x_{j-1} = \gamma + e_{j-1} \qquad (20)$$

From (19) and (20), we have

$$e_{j+1} = e_j - \frac{(e_j - e_{j-1}) f(\gamma + e_j)}{f(\gamma + e_j) - f(\gamma + e_{j-1})} \qquad (21)$$

Now expanding $f(\gamma + e_j)$by Taylor's theorem, we have

$$f(\gamma + e_j) = f(\gamma) + \frac{e_j}{1!} f'(\gamma) + \frac{e_j^2}{2!} f''(\gamma) + \cdots$$

Since γ is the zero of $f(x) = 0$, As $f(\gamma) = 0$, we have

$$f(\gamma + e_j) = \frac{e_j}{1} f'(\gamma) + \frac{e_j^2}{2} f''(\gamma) + \cdots \qquad (22)$$

Again the denominator of (21) is

$$f(\gamma + e_j) - f(\xi + e_{j-1}) = \left[f(\gamma) + \frac{e_j}{1!} f'(\gamma) + \frac{e_j^2}{2!} f''(\gamma) + \cdots\right] - \left[f(\gamma) + \frac{e_{j-1}}{1!} f'(\gamma) + \frac{e_{j-1}^2}{2!} f''(\gamma) + \cdots\right]$$

$$f(\gamma + e_j) - f(\gamma + e_{j-1}) = (e_j - e_{j-1}) f'(\gamma) + \frac{(e_j^2 - e_{j-1}^2)}{2} f''(\gamma) + \cdots \qquad (23)$$

Using (22) and (23) equation (21) becomes

$$e_{j+1} = e_j - \frac{[e_j f'(\gamma) + \frac{e_j^2}{2} f''(\gamma) + \cdots]}{[f'(\gamma) + \frac{(e_j + e_{j-1})}{2} f''(\gamma) + \cdots]}$$

Dividing nominator and denominator by $f'(\gamma)$, we have

$$e_{j+1} = e_j - \left[e_j + \frac{e_j^2}{2}\frac{f''(\gamma)}{f'(\gamma)} + \cdots\right]\left[1 + \frac{(e_j + e_{j-1})}{2}\frac{f''(\gamma)}{f'(\gamma)} + \cdots\right]^{-1}$$

Or

$$e_{j+1} = e_j e_{j-1}\frac{f''(\gamma)}{2f'(\gamma)} + O(e_j^2 e_{j-1} + e_j e_{j-1}^2)$$

$$e_{j+1} = k\, e_j e_{j-1} \qquad (24)$$

Where $\frac{f''(\gamma)}{2f'(\gamma)} = k$ constant and higher power of is e_j neglected and where (24) is error equation.

Let β be the rate (order) of convergence, then by the definition, we have

$$e_j = e_{j-1}^{\beta} \cdot k' \qquad (25)$$

$$\Rightarrow \quad \frac{e_j}{e_{j-1}^{\beta}} = k' \Rightarrow \quad e_{j-1} = \left(\frac{e_j}{k'}\right)^{1/\beta}$$

Equation (24) becomes

$$e_{j+1} = e_j \cdot k \frac{(e_j)^{1/\beta}}{(k')^{1/\beta}}$$

$$e_{j+1} = e_j^{1+1/\beta} \cdot \frac{k}{(k')^{1/\beta}} \qquad (26)$$

Similarly

$$e_{j+1} = e_j^{\beta} k'$$

Therefore equation (26) becomes

$$e_j^{\beta} \cdot k' = e_j^{1+1/\beta} \cdot \frac{k}{(k')^{1/\beta}} \qquad (27)$$

Equating power of e_j both sides

$$\Rightarrow \quad 1 + 1/\beta = \beta \quad or \quad \beta^2 - \beta - 1 = 0$$

$$\Rightarrow \quad \beta = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Choosing +ve sign, we have

$$\beta = \frac{1+\sqrt{5}}{2} = \frac{3.236}{2} = 1.618$$

This is the order of convergence of secant method and the convergence is referred to as superliner convergence. We note that, this method fails if at any iteration$f(x_j) = f(x_{j-1})$, and show that it does not converge.

Muller Method

Muller method is an iterative method in which do not require derivative of the function. Muller method is beneficial in evaluating the roots of polynomials. It is a similarity of the secant method. In this method the function $f(x)$ is approximated of the root. Let as assume for $f(x)$ a polynomial of second degree is given by

$$f(x) = a_0(x - x_j)^2 + a_1(x - x_j) + a_2 \qquad (28)$$

Substituting $x = x_j, x_{j-1}$ *and* x_{j-2}. Let $f(x_j) = f_j$, $f(x_{j-1}) = f_{j-1}$ and $f(x_{j-2}) = f_{j-2}$, determine a_0 ,a_1 and a_2, then we have

$$f_j = a_0(x_j - x_j)^2 + a_1(x_j - x_j) + a_2 = a_2 \qquad (29)$$

$$f_{j-1} = a_0(x_{j-1} - x_j)^2 + a_1(x_{j-1} - x_j) + a_2 \qquad (30)$$

$$f_{j-2} = a_0(x_{j-2} - x_j)^2 + a_1(x_{j-2} - x_j) + a_2 \qquad (31)$$

From equations (29)-(31), we get

$$a_2 = f_j \qquad (32)$$

$$a_1 = \frac{1}{D}\left[(x_j - x_{j-2})^2(f_j - f_{j-1}) - (x_j - x_{j-1})^2(f_j - f_{j-2})\right] \qquad (33)$$

$$a_0 = \frac{1}{D}\left[(x_j - x_{j-2})(f_j - f_{j-1}) - (x_j - x_{j-1})(f_j - f_{j-2})\right] \qquad (34)$$

Where

$$D = (x_{j-1} - x_j)^2(x_{j-2} - x_j) - (x_{j-2} - x_j)^2(x_{j-1} - x_j)$$

$$D = (x_j - x_{j-1})(x_j - x_{j-2})(x_{j-1} - x_{j-2}) \qquad (35)$$

Solving the equation (28) for $(x - x_j)$ and taking $x$by x_{j+1}, we get

$$x_{j+1} = x_j - \frac{2a_2}{a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}, \qquad j = 2,3, \ldots \qquad (36)$$

The sign in the denominator is selected so that the denominator becomes largest in magnitude.

Generally, Muller method of iteration converges quadratically almost for all initial approximations. If no better approximations are known, we can put $x_{j-2} = -1, x_{j-1} = 0$and $x_j = 1$.

Rate (order) of convergence of Muller method

Let any number γ for $f(\gamma) = 0$ is exact root of equation $f(x) = 0$, on substituting, $x_j = \gamma + e_j$, $x_{j-1} = \gamma + e_{j-1}$ and $x_{j-2} = \gamma + e_{j-2}$

From (35), we have

$$\Rightarrow \quad D = (\gamma + e_j - \gamma - e_{j-1})(\gamma + e_j - \gamma - e_{j-2})(\gamma + e_{j-1} - \gamma - e_{j-2})$$

$$D = (e_j - e_{j-2})(e_j - e_{j-1})(e_{j-1} - e_{j-2}) \qquad (37)$$

From (33), we get

$\Rightarrow \quad a_2 = f(\gamma + e_j)$

Expanding $f(\gamma + e_j)$ in Taylors series

$$a_2 = f(\gamma) + \frac{e_j}{1!} f'(\gamma) + \frac{e_j^2}{2!} f''(\gamma) + \frac{e_j^3}{3!} f'''(\gamma) + \cdots$$

Since γ is the zero of $f(x) = 0$. As $f(\gamma) = 0$, we have

$$a_2 = e_j f'(\gamma) + \frac{e_j^2}{2} f''(\gamma) + \frac{e_j^3}{6} f'''(\gamma) + \cdots \quad (38)$$

From (33), we get

$$\Rightarrow \quad a_1 = \frac{1}{D}[(\gamma + e_j - \gamma - e_{j-2})^2 \{f(\gamma + e_j) - f(\gamma - e_{j-1}\} - (\gamma + e_j - \gamma - e_{j-1})^2 \times \{f(\gamma + e_j) - f(\gamma + e_{j-2})\}]$$

$$a_1 = \frac{1}{D}[(e_j - e_{j-2})^2 \{f(\gamma + e_j) - f(\gamma - e_{j-1}\} - (e_j - e_{j-1})^2 \{f(\gamma + e_j) - f(\gamma + e_{j-2})\}]$$

Since γ is the zero of $f(x) = 0$. As $f(\gamma) = 0$, we have

$$a_1 = \frac{1}{D}[(e_j - e_{j-2})^2 \{\left(e_j f'(\gamma) + \frac{e_j^2}{2} f''(\gamma) + \frac{e_j^3}{6} f'''(\gamma) + \cdots\right) - (e_{j-1} f'(\gamma) + \frac{e_{j-1}^2}{2} f''(\gamma) + \frac{e_{j-1}^3}{6} f'''(\gamma) + \cdots)\} - (e_j - e_{j-1})^2 \times \{(e_j f'(\gamma) + \frac{e_j^2}{2} f''(\gamma) + \frac{e_j^3}{6} f'''(\gamma) + \cdots) - (e_{j-2} f'(\gamma) + \frac{e_{j-2}^2}{2} f''(\gamma) + \frac{e_{j-2}^3}{6} f'''(\gamma) + \cdots)\}]$$

$$a_1 = \frac{1}{D}[(e_j - e_{j-2})(e_j - e_{j-1})(e_{j-1} - e_{j-2}) f'(\gamma) + (e_j - e_{j-2})(e_j - e_{j-1})(e_{j-1} - e_{j-2}) \times e_j f''(\gamma) + (e_j - e_{j-2})(e_j - e_{j-1})(e_{j-1} - e_{j-2}) \frac{1}{6}\{2e_j^2 + e_j e_{j-1} + e_j e_{j-2} - e_{j-1} e_{j-2}\} f'''(\gamma) + \cdots]$$

From (37), we have

$$a_1 = f'(\gamma) + e_j f''(\gamma) \frac{1}{6}\{2e_j^2 + e_j e_{j-1} + e_j e_{j-2} - e_{j-1} e_{j-2}\} f'''(\gamma) + \cdots$$

From (34), we get

$$\Rightarrow \quad a_0 = \frac{1}{D}[(e_j - e_{j-2})\{(f(\gamma) + \frac{e_j}{1!} f'(\gamma) + \frac{e_j^2}{2!} f''(\gamma) + \frac{e_j^3}{3!} f'''(\gamma) + \cdots) - (f(\xi) + \frac{e_{j-1}}{1!} f'(\xi) + \frac{e_{j-1}^2}{2!} f''(\xi) + \frac{e_{j-1}^3}{3!} f'''(\xi) + \cdots)\} - (e_j - e_{j-1})\{(f(\gamma) + \frac{e_j}{1!} f'(\gamma) + \frac{e_j^2}{2!} f''(\gamma) + \frac{e_j^3}{3!} f'''(\gamma) + \cdots) - (f(\xi) + \frac{e_{j-2}}{1!} f'(\xi) + \frac{e_{j-2}^2}{2!} f''(\gamma) + \frac{e_{j-2}^3}{3!} f'''(\gamma) + \cdots)\}]$$

Since γ is the zero of $f(x) = 0$. As $f(\gamma) = 0$, we have

$$a_0 = \frac{1}{D}[\frac{1}{2}(e_j - e_{j-2})\{(e_j - e_{j-1})(e_j - e_{j-2}) f''(\gamma) + \frac{1}{6}(e_j - e_{j-2})(e_j - e_{j-1}) \times \{e_j(e_{j-1} - e_{j-2})(e_j^2 - e_{j-2}^2)\} f'''(\gamma) + \cdots]$$

$$a_0 = \frac{1}{2} f''(\gamma) + \frac{1}{6}(e_j + e_{j-1} + e_{j-2}) f'''(\gamma) + \cdots$$

Now we find

$$a_1^2 - 4a_0 a_2 = [f'(\gamma)]^2 + 2e_j f'(\gamma) f''(\gamma) + e_j^2 [f''(\xi)]^2 + \frac{1}{3}[2e_j^2 + e_j e_{j-1} + e_j e_{j-2} - e_{j-1} e_{j-2}] f'(\gamma) f'''(\gamma) + \cdots - 4\left[e_j f'(\gamma) + \frac{1}{2} e_j^2 f''(\gamma) \frac{1}{6} e_j^2 f'''(\gamma) + \cdots\right] \times [\frac{1}{2} f''(\gamma) + \frac{1}{6}(e_j + e_{j-1} + e_{j-2}) f'''(\gamma) + \cdots]$$

$$= [f'(\gamma)]^2 - \frac{1}{3}(e_j e_{j-1} + e_j e_{j-2} + e_{j-1} e_{j-2}) f'(\gamma) f'''(\gamma) + \cdots$$

$$\sqrt{a_1^2 - 4a_0 a_2} = f'(\gamma)[1 - \frac{1}{6}(e_j e_{j-1} + e_j e_{j-2} + e_{j-1} e_{j-2}) k_3 + \cdots]$$

Where $k_i = \frac{f^{(i)}(\gamma)}{f'(\gamma)}$, $i = 2,3, \ldots$

$$a_1 + \sqrt{a_1^2 - 4a_0 a_2} = 2f'(\gamma)\left[1 + \frac{1}{2} e_j k_2 + \frac{1}{6}(e_j^2 - e_{j-1} e_{j-2}) k_3 + \cdots\right]$$

Hence, we obtain from (36), we have

$$e_{j+1} = e_j - [e_j + \frac{1}{2} e_j^2 k_2 + \frac{1}{6} e_j^3 k_3 + \cdots][1 + \{\frac{1}{2} e_j k_2 + \frac{1}{6}(e_j^2 - e_{j-1} e_{j-2}) k_3 + \cdots]^1$$

$$= e_j - [e_j + \frac{1}{2} e_j^2 k_2 + \frac{1}{6} e_j^3 k_3 + \cdots][1 - \frac{1}{2} e_j k_2 + \frac{1}{4} e_j^2 k_2^2 - \frac{1}{6}(e_j^2 - e_{j-1} e_{j-2}) k_3 + \cdots]$$

$$= \frac{1}{6} e_j e_{j-1} e_{j-2} k_3 + \cdots$$

Therefore, the error equation associated with the Muller method is given by

$$e_{j+1} = k e_{j-2} e_{j-1} e_j \quad (39)$$

Where

$$k = \frac{1}{6} k_3 = \frac{1}{6} \frac{f'''(\gamma)}{f'(\gamma)} \quad (40)$$

We now seek a relation of the form

$$e_{j+1} = k' e_j^\beta \quad (41)$$

Where k' and β are to be determined. From (41) we get

$$e_j = k' e_{j-1}^\beta, \ or \ e_{j-1} = k'^{-1/\beta} e_j^{1/\beta}$$

$$e_{j-1} = k' e_{j-2}^\beta, \ or \ e_{j-2} = k'^{-1/\beta} e_{j-1}^{1/\beta} = k'^{-(1/\beta + 1/\beta^2)} e_j^{1/\beta^2}$$

Substituting the value of e_{j+1}, e_{j-1} and e_{j-2} in (39), we have

$$e_j^\beta = k k'^{-(1 + \frac{2}{\beta} + \frac{1}{\beta^2})} e_j^{1 + \frac{1}{\beta} + \frac{1}{\beta^2}} \quad (42)$$

Comparing the powers of e_j on both sides, we get

$$\beta = 1 + \frac{1}{\beta} + \frac{1}{\beta^2}$$

Or

$$F(\beta) = \beta^3 - \beta^2 - \beta - 1 = 0 \quad (43)$$

The equation $F(\beta) = 0$ has the smallest positive zero of the interval (1,2), we use the N-R method to determine this root, we get

$$\beta_{j+1} = \beta_j - \frac{F(\beta_j)}{F'(\beta_j)} = \beta_j - \frac{F(\beta_j^3 - \beta_j^2 - \beta_j - 1)}{F(3\beta_j^2 - 2\beta_j - 1)}$$

Or

$$\beta_{j+1} = \frac{2\beta_j^3 - \beta_j^2 + 1}{3\beta_j^2 - 2\beta_j^2 - 1}, \quad j = 0,1, \ldots$$

Starting $\beta_0 = 2$, we get

$\beta_1 = 1.8571, \beta_2 = 1.8395, \beta_3 = 1.8393, \ldots$

Therefore, the root of the equation (43) is $\beta = 1.84$ approximately. Where, the rate of convergence of this method is 1.84.

Numerical Experiments

In this part we are going to select some examples and realize the number of iteration that is needed for the given precision. We will apply in $e = 0.1^{10}$.

Example I. $f(x) = x^3 - 2x - 5 = 0,\ [2,3]$

Example II. $f(x) = xe^x - 1, [-1,1]$

Example III. $f(x) = \frac{1}{x} - \sin(x) + 1 = 0, [-1.3, -0.5]$

In the Muller method for the example I, we take initial approximations as $x_0 = 3$, $x_1 = 2$and $x_2 = 1$, for example II, initial approximations as $x_0 = 1$, $x_1 = -1$and $x_2 = -6$ and for example III initial estimatesare as $x_0 = -0.5$, $x_1 = -1.3$ and $x_2 = -4$. The results of the examples I-III aregiven in table 1.

Table I. Comparison of Examples.

Equation	initial value (x_0)	Number of iteration					Root
		Bisection	Regula-Falsi	Newton	Secant	Muller	
$x^3 - 2x - 5$	3	34	24	5	7	5	2.0945514815
$xe^x - 1$	1	35	22	5	14	7	0.5671432904
$\frac{1}{x} - \sin(x) + 1$	-0.5	33	14	4	9	6	-0.6294464841

CONCLUSION

In this paper, we defined different forms of non-linear equations and stated a number of methods to find the roots of such equations and also we discussed the procedure of convergent of some iterative methods. The bisection method is slow but steady. It is, however, the simplest method and it is never fails. If the evaluation of $f(x)$is rapid, then the use of the bisection is strongly advised. The method of Regula-Falsi is slow and it is first-order convergent. Most often, it is found superior to the bisection method. The secant method is faster than the (R-F) method. The most commonly used method is the Newton-Raphson method, once the initial value of the root has been found near to the actual root, the convergence of this method is faster. On comparing the above five methods, we conclude that Newton-Raphson and Muller methods have less number of iteration, so these are more efficient. Further, Newton-Raphson has the order of convergence 2, which is the greatest of all four. Hence Newton-Raphson is most effective out of these five methods.

Acknowledgment

First of all I would like to express my gratitude to my supervisor**Dr. KuldipKatiyar**. He helped me a lot in my research work, and without his help it was impossible.I also thank the Mathematics Department for providing me the opportunity to work on my research. They were really supportive throughout my research work.

References

1. Curtis F. Gerald and Patrick O. Whealty, "Applied Numerical Analysis", 7th Edition, Pearson Education, Inc, (2004).
2. Walter Gautschi,"Numerical Analysis",2nd Edition, Springer Science+Business Media, LLC, New York, (2010).
3. J.B. Dixit, "Numerical Method", 1st Edition, University science press New Delhi, (2010).
4. Babu Ram, "Numerical method", Dorling Kindersley (India) Pvt.Ltd, (2010).
5. Mukesh Grover and S.S. chuhan, "Numerical Method (Numerical Analysis) with programing in C, C++" 1st Edition, Teach India publications New Delhi, (2010).
6. E Balagurusamy, "Numerical Methods", Tata McGraw Hill Education Private Limited New Delhi, (2011).
7. Richard L. Burden, and J. Douglas Faires, "Numerical Analysis", 9th edition, Richard Stsratton publisher, (2011).
8. M.K. Jain, S.R.K. Iyengar, R.K.Jain, "Numerical Methods for Scientific and Engineering Computation", 6th edition, New Age International Publishers, (2012).
9. Steven C. Chapra, "Applied Numerical Methods *with MATLAB®for Engineers and Scientists", 3rd*Edition, McGraw-Hill Companies, Inc., (2012).
10. Timothy Sauer., "Numerical analysis", 2nd Edition, Pearson Education, Inc., (2012).
11. SantanuSaha Ray, "Numerical Analysis with Algorithms and Programming", Taylor & Francis Group, LLC. (2016).
12. Noor, M.A. and Ahmad, F., 2006. Numerical comparison of iterative methods for solving nonlinear equations. *Applied mathematics and computation*, *180*(1), pp.167-172.
13. Noor, M.A., Ahmad, F. and Javeed, S., 2006. Two-step iterative methods for nonlinear equations. *Applied Mathematics and Computation*, *181*(2), pp.1068-1075.
14. Naghipoor, J., Ahmadian, S. A., &Soheili, A. R. (2008). An improved regulafalsi method for finding simple zeros of nonlinear equations. *Applied Mathematical Sciences*, *2*(8), 381-386.
15. Shaw, S. and Mukhopadhyay, B., 2015. An improved Regulafalsi method for finding simple roots of nonlinear equations. *Applied Mathematics and Computation*, *254*, pp.370-374.
16. Parida, P.K. and Gupta, D.K., 2006. An improved regula-falsi method for enclosing simple zeros of nonlinear equations. *Applied mathematics and computation*, *177*(2), pp.769-776.
17. Chen, J. and Li, W., 2006. An exponential regulafalsi method for solving nonlinear equations. *Numerical algorithms*, *41*(4), pp.327-338.
18. Chen, J. and Li, W., 2007.An improved exponential regulafalsi methods with quadratic convergence of both diameter and point for solving nonlinear equations. *Applied numerical mathematics*, *57*(1), pp.80-88.
19. Suhadolnik, A., 2012.Combined bracketing methods for solving nonlinear equations. *Applied Mathematics Letters*, *25*(11), pp.1755-1760.
20. Suhadolnik, A., 2013. Superlinear bracketing method for solving nonlinear equations. *Applied Mathematics and Computation*, *219*(14), pp.7369-7376.

****