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International Journal of Recent Scientific Research
Vol. 6, Issue, 5, pp.3826-3833, May, 2015

International Journal of Recent Scientific Research

ISSN: 0976-3031

RESEARCH ARTICLE

ON SECOND – ORDER DIFFERENTIAL SUBORDINATION AND SUPERORDINATION OF ANALYTIC AND MULTIVALENT FUNCTIONS

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ARTICLE INFO

Article History:

Received 14th, April, 2015
Received in revised form 23th, April, 2015
Accepted 13th, May, 2015
Published online 28th, May, 2015

Key words:

Analytic functions, differential subordination, superordination multivalent function, Hadamard product (convolution).

ABSTRACT

In this paper, we give some results for differential subordination and superordination for analytic and multivalent functions in the open unite disk U involving linear operator $F_{\lambda,p}^{m}(f*g)(z)$. These results are obtained by investigating appropriate classes of admissible functions.

INTRODUCTION

Preliminaries

Let $H(U)$ denote the class of analytic functions in the open unit disk $U = \{z: z \in \mathbb{C}\ and\ |z| < 1\}$ and let $H[a,n]$ denote the subclass of $H(U)$ of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$,where $a \in \mathbb{C}$ and $n \in N$with $H_0 = H[0,1]$ and $H_1 = H[1,1]$.If f, g are members of $H(U)$ we say that a function f is subordinate to a function g or g is said to be superordinate to f if there exists a Schwarz function $w(z)$ which is analytic in U, with $w(0) = 0$, $|w(z)| < 1$ for all $(z \in U)$, such that $f(z) = g(w(z))$. In such a case we write $f \prec g$. Further, if the function g is univalent in U then we have the following equivalent,(see [3,9]).

$f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $\mathcal{A}(p)$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (z \in U,\ p \in N = \{1,2,3, \dots\}). \tag{1.1}$$

For function $g \in \mathcal{A}(p)$ given by $g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, (p \in N)$, the Hadmard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z).$$

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For function $f,g \in \mathcal{A}(p)$, we defined the linear operator$F_{\lambda,p}^{m}: \mathcal{A}(p) \to \mathcal{A}(p)$ $(\lambda \geq 0, m \in N_0 = N \cup \{0\})$ by :$F_{\lambda,p}^{0}(f*g)(z) = (f*g)(z)$,

$$F_{\lambda,p}^{1}(f*g)(z) = F_{\lambda,p}(f*g)(z) = (1-\lambda)(f*g)(z) + \frac{\lambda z}{p}\left((f*g)(z)\right)'$$

$$= z^p + \sum_{k=p+1}^{\infty} \frac{p+\lambda(k-p)}{p} a_k b_k z^k,$$

and

$$F_{\lambda,p}^{2}(f*g)(z) = F_{\lambda,p}\left[F_{\lambda,p}(f*g)(z)\right],$$

therefore, it can be easily seen that

$$F_{\lambda,p}^{m}(f*g)(z) = F_{\lambda,p}\left(F_{\lambda,p}^{m-1}(f*g)(z)\right)$$

$$= z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+\lambda(k-p)}{p}\right)^m a_k b_k z^k, (\lambda \geq 0). \quad (1.2)$$

From (1.2) we can easily deduce that

$$\frac{\lambda z}{p}\left(F_{\lambda,p}^{m}(f*g)(z)\right)' = F_{\lambda,p}^{m+1}(f*g)(z) - (1-\lambda)F_{\lambda,p}^{m}(f*g)(z), \ (\lambda > 0). \quad (1.3)$$

The operator $F_{\lambda,p}^{m}(f*g)$ was introduced and studied by Selvaraj and Selvakumaran [14], Aouf and Mostafa [2] and for $= 1$, was introduced by Aouf and Mostafa [1].

Remark

1. Taking $m = 0$ and $b_k = \frac{(\alpha_1)_{k-1}\cdots\cdots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\cdots\cdots(\beta_s)_{k-1}(1)_{k-1}}$ $(\alpha_i, \beta_j \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}, (i = 1,2 \ldots q), (j = 1,2 \ldots s)$,
2. $q \leq s+1, q, s \in N_0$ in (1.2), the operator $F_{\lambda,p}^{m}(f*g)$ reduces to the Dziok-Srivastava operator $H_{p,q,s}$ (α_1) which generalized many other operator (see [6]).
3. Taking $m = 0$ and $b_k = \frac{p+l+\lambda(k-p)}{p+l}$ $(\lambda > 0\,;\, p \in N\,;\, l, n \in N_0)$ in (1.2), the operator $F_{\lambda,p}^{m}(f*g)$ reduces to Catas operator $I_p^n(l,\lambda)$ which generalizes many other operators (see [4]).
4. The method of differential subordinations (also known as the admissible functions method) was introduced by Miller and Mocanu [7,8] and developed in [9,10].
5. Let Ω and Δ be any sets in $\mathbb{C}$ and let p be an analytic function in the unit disk$U$ with $p(0) = a$ and let $\psi\,(r,s,t;z): \mathbb{C}^3 \times U \to \mathbb{C}$. The heart of this theory deals with generalizations of the following implication :
6. $\{\psi(p(z), zp'(z), z^2p''(z); z), (z \in U)\} \subset \Omega \Rightarrow p(U) \subset \Delta$. In [10] the authors introduce the dual problem of the differential subordination which they call differential superordination.
7. $\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z), (z \in U)\} \Rightarrow \Delta \subset p(U)$.
8. **Definition 1.1** [9] Let $\psi: \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second – order) differential subordination,
9. $\{\psi(p(z), zp'(z), z^2p''(z); z), (z \in U)\} \prec h(z)$, then p is called a solution of differential subordination. The univalent function q is called a dominant, if $p \prec q$ for all p satisfying (iii).
10. A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (iii) is said to be the best dominant of (iii).
11. **Definition 1.2** [10] Let $\psi: \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be analytic in U. If p and $\psi(p(z), zp'(z), z^2p''(z); z)$ are univalent in U and satisfy the (second – order) differential superordination.
12. $h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z)$, then $p$is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination,, or more simply a subordinate if $q \prec p$ for all p satisfying (iv). A univalent subordinant$\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants$q$ of (iv) is said to be the best subordinant. (Note that the best subordinant is unique up to a rotation of U). For Ω a set in $\mathbb{C}$, with ψ and p as given in Definition 1.2, suppose (iv) is replaced by
13. $\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z), (z \in U)\}$.

To prove our results, we need the following definitions and Lemmas.

Definition 1.3 [9] Denote by Q the set of all functions q that are analytic and injective on $\overline{U}\backslash E(q)$, where

$E(q) = \left\{\xi \in \partial U : \lim_{z\to\xi} q(z) = \infty\right\},$
and are such that $q^{'}(\xi) \neq 0$ for $\xi \in \partial U \backslash E(q)$. Further let the subclass of Q for which $q(0) = a$ be denoted by $Q(a), Q(0) \equiv Q_0$ and $Q(1) = Q_1$.

Definition 1.4 [9] Let Ω be a set in $\mathbb{C}$; $q \in Q$ and n be appositive integer. The class of admissible functions$\Psi_n[\Omega, q]$ consists of those functions $\psi: \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissimbilty condition:

$\psi(r,s,t;z) \notin \Omega,$

whenever $r = q(\xi), s = k\xi q^{'}(\xi),$

$$Re\left\{\frac{t}{s}+1\right\} \geq kRe\left\{1+\frac{\xi\, q^{''}(\xi)}{q'(\xi)}\right\},$$

where $z \in U, \xi \in \partial U \backslash E(q)$ and $k \geq n$. We write $\Psi'_1[\Omega, q] = \Psi[\Omega, q]$.

In particular, when $q(z) = M\frac{Mz+a}{M+\bar{a}z}$, with $M > 0$ and $|a| < M$, then $q(U) = U_M = \{w: |w| < M\}, q(0) = a, E(q) = \emptyset$ and $q \in Q$. In this case, we set $\Psi_n[\Omega, M, a] = \Psi[\Omega, q] = \Psi_1[\Omega, q]$, and in the special case when the set $Q = U_M$, the class is simply denoted by $\Psi_n[M, a]$.

Definition 1.5[10] Let Ω be a set in $\mathbb{C}$, $q(z) \in H[a, n]$ with $q^{'}(z) \neq 0$. The class of admissible functions $\Psi^{'}[\Omega, q]$ consist of this functions ψ : $\mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:

$\psi(r,s,t;\xi) \in \Omega,$

whenever $r = q(z), s = \frac{zq^{'}(z)}{b},$

$$Re\left\{\frac{t}{s}+1\right\} \leq \frac{1}{b}Re\left\{1+\frac{zq^{''}(z)}{q'(z)}\right\},$$

when $z \in U, \xi \in \partial U$ and $b \geq n \geq 1$. In particular, we write $\Psi_1^{'}[\Omega, \mathrm{q}] = \Psi^{'}[\Omega, \mathrm{q}]$.

Lemma 1.1 [9] Let $\psi \in \Psi_n[\Omega, \mathrm{q}]$ with $q(0) = a$. If the analytic function $g(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots$, satisfies

$\psi(g(z), zg^{'}(z), z^2 g^{''}(z);\ z) \in \Omega,$
Then

$g(z) \prec q(z), \ \ (z \in U).$

Lemma 1.2 [10] Let $\psi \in \Psi_n^{'}[\Omega, q]$ with $q(0) = a$, $g \in Q(a)$ and $\psi(g(z), zg^{'}(z), z^2 g^{''}(z);\ z)$ is univalent in U, then

$\Omega \subset \{\psi(g(z), zg^{'}(z), z^2 g^{''}(z); z), (z \in U)\},$

implies

$q(z) \prec g(z)$

In fact, the study of the class of admissible functions was revived recently by Mustafa and Darus [11] and Cho [5]. A similar problem for analytic functions was studied by many others for example see [10,12,13]

The object of the present paper, we give some results for differential subordination and superordination for multivalent function involving the linearoperator $F_{\lambda,p}^{m}(f * g)(z)$.

Differential suboridination results associated with linear operator

Difintion 2.1 Let Ω be a set in $\mathbb{C}$, $q \in Q_0 \cap H[0, p]$. The class of admissible functions $\Psi_n[\Omega, \mathrm{q}]$ consists of those functions ϕ : $\mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:

$\phi(u, v, w\,; z, \xi) \notin \Omega$

whenever

$$u = q(\xi), v = \frac{k\xi q'(\xi) + \frac{p(1-\lambda)}{\lambda} q(\xi)}{\frac{p}{\lambda}}$$

And

$$Re\left\{\frac{p^2 w + 2p^2(1-\lambda)v - 3p^2(1-\lambda)^2 u}{\lambda p v - p\lambda(1-\lambda)u}\right\} \geq kRe\left\{1 + \frac{\xi q''(\xi)}{q'(\xi)}\right\},$$

where $z \in U$, $\xi \in \partial U \backslash E(q), \lambda > 0$ *and* $k \geq p$.

Theorem 2.1 Let $\phi \in \Psi_n[\Omega, q]$. If $f \in \mathcal{A}(p)$ satisfies

$$\phi\left(F_{\lambda,p}^{m}(f*g)(z), F_{\lambda,p}^{m+1}(f*g)(z), F_{\lambda,p}^{m+2}(f*g)(z)\right) \subset \Omega, \quad (2.1)$$

where $\lambda > 0, m \in N_0 = \{0,1,2 \ldots\}, z \in U$.

Then
$F_{\lambda,p}^{m}(f*g)(z) \prec q(z), (z \in U)$.

Proof. By using (1.2) and (1.3), we get the equivalent relation

$$F_{\lambda,p}^{m+1}(f*g)(z) = \frac{z\left(F_{\lambda,p}^{m}(f*g)(z)\right)' + \frac{p(1-\lambda)}{\lambda} F_{\lambda,p}^{m}(f*g)(z)}{\frac{p}{\lambda}} \quad (2.2)$$

Assum that

$$G(z) = F_{\lambda,p}^{m}(f*g)(z). \quad (2.3)$$

Then

$$F_{\lambda,p}^{m+1}(f*g)(z) = \frac{zG'(z) + \frac{p(1-\lambda)}{\lambda} G(z)}{\frac{p}{\lambda}}. \quad (2.4)$$

Further computation show that

$$F_{\lambda,p}^{m+2}(f*g)(z) = \frac{z^2 G''(z) + \left(1 + \frac{2p(1-\lambda)}{\lambda}\right) zG'(z) + \frac{p^2(1-\lambda)^2}{\lambda^2} G(z)}{\frac{p^2}{\lambda^2}}. \quad (2.5)$$

Define the transformation from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, v = \frac{s + \frac{p(1-\lambda)}{\lambda} r}{\frac{p}{\lambda}}, w = \frac{t + (1 + \frac{2p(1-\lambda)}{\lambda})s + \frac{p^2(1-\lambda)^2}{\lambda^2} r}{\frac{p^2}{\lambda^2}} \quad (2.6)$$

Let

$$\psi(r,s,t;z) = \emptyset(u,v,w;z) = \phi\left(r, \frac{s + \frac{p(1-\lambda)}{\lambda} r}{\frac{p}{\lambda}}, \frac{t + (1 + \frac{2p(1-\lambda)}{\lambda})s + \frac{p^2(1-\lambda)^2}{\lambda^2} r}{\frac{p^2}{\lambda^2}}\right). \quad (2.7)$$

The proof shall make use of Lemma 1.1 using equations (2.3), (2.4) and (2.5), we obtain

$$\psi(G(z), zG'(z), z^2G''(z); z) = \phi\left(F_{\lambda,p}^{m}(f*g)(z), F_{\lambda,p}^{m+1}(f*g)(z), F_{\lambda,p}^{m+2}(f*g)(z), z\right) (2.8)$$

Therefore, by making use (2.1), we get

$$\psi(G(z), zG'(z), z^2G''(z); z) \in \Omega. \quad (2.9)$$

The proof is completed if it can be show, that the admissibility condition for $\phi \in \Psi_n[\Omega, q]$ is equivalent to the admissibility for ψ as given in Definition 1.4. Note that

$$\frac{t}{s}+1=\frac{p^2w+2p^2(1-\lambda)r-3p^2(1-\lambda)^2u}{\lambda pv-p\lambda(1-\lambda)u}\ , \qquad (2.10)$$

and hence $\psi \in \Psi_n[\Omega. q]$. By Lemma 1.1,
$G(z) \prec q(z)$,
or
$F_{\lambda,\rho}^{m}(f*g)(z) \prec q(z)$.

We consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(U)$, where h is a conformal mapping of U onto Ω and the class is written as $\Psi_n[h, q]$. The following results is an immediate consequence of Theorem 2.1.

Theorem 2.2 Let $\phi \in \Psi_n[h, q]$. If $f \in \mathcal{A}(p)$ satisfies

$$\phi\big(F_{\lambda,p}^{m}(f*g)(z)\,,F_{\lambda,p}^{m+1}(f*g)(z)\,,F_{\lambda,p}^{m+2}(f*g)(z)\,;z\big) \prec h(z) \qquad (2.11)$$

where $\lambda > 0\,, m \in N_0\,, z \in U$.
Then
$F_{\lambda,p}^{m}(f*g)(z) \prec q(z), \quad (z \in U)$.

The next result is an extension of Theorem 2.2 to the case where the behavior of $q(z)$ on ∂U is unknown.

Corollary 2.1 Let $\Omega \subset \mathbb{C}, q$ be univalent in U and $q(0) = 0$. Let $\phi \in \Psi_n[\Omega, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{A}(p)$ and satisfies

$$\phi\big(F_{\lambda,p}^{m}(f*g)(z), F_{\lambda,p}^{m+1}(f*g)(z)\,,F_{\lambda,p}^{m}(f*g)(z)\,;z\big) \in \Omega,$$

where $\lambda > 0\,, m \in N_0$ *and* $z \in U$.
Then
$F_{\lambda,p}^{m}(f*g)(z) \prec q(z), (z \in U)$.

Proof From Theorem 2.1 yields $F_{\lambda,p}^{m}(f*g)(z) \prec q(\rho z)$.The result now deduced from $q_\rho(z) \prec q(z)$.

Theorem 2.3 Let h and q be univalent in U, with $q(0) = 0$ and set$q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi :\ \mathbb{C}^3 \times U \to \mathbb{C}$ satisfy one of the following conditions:

1. $\phi \in \Psi_n[h, q_\rho]$, for some $\rho \in (0,1)$.
2. There exists $\rho_0 \in (0,1)$ such that $\phi \in \Psi_n[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$. If $f \in \mathcal{A}(p)$ and satisfies (2.11), then

$F_{\lambda,p}^{m}(f*g)(z) \prec q(z)$.

Proof Case (1) : By using Theorem 2.1, we obtain $F_{\lambda,p}^{m}(f*g)(z) \prec q_\rho(z)$, since $q_\rho(z) \prec q(z)$, we deduce

$F_{\lambda,p}^{m}(f*g)(z) \prec q(z)$.

Case (2): Let $G(z) = F_{\lambda,p}^{m}(f*g)(z)$ and $G_\rho(z) = G(\rho z)$.

Then$\phi\big(G_\rho(z)\,,zG'_\rho(z)\,,z^2G''_\rho(z);\ \rho z\big) = \ \phi(G(\rho z), zG'(\rho z), z^2G''(\rho z);\ \rho z) \in h_\rho(U)$. By using Theorem 2.1 and the comment associated with $\phi(G(z)\,,zG'(z)\,,z^2G''(z);\ w(z)) \in \Omega$,where w is any mapping U in to U , with $w(z) = \rho z$, we obtain $G_\rho(z) = q_\rho(z)$ for $\rho \in (\rho_0, 1)$. By letting $\rho \to 1^-$, we get $G(z) \prec q(z)$.

Therefore,

$F_{\lambda,p}^{m}(f*g)(z) \prec q(z)$.

The next result gives the best dominant of the differential subordination (2.11).

Theorem 2.4 Let h be univalent in U and let $\phi :\ \mathbb{C}^3 \times U \to \mathbb{C}$. Suppose that the differential equation

$$\phi(q(z), z\,q'(z), z^2 q''(z);\ z) = h(z) \tag{2.12}$$

has a solution q with $q(0) = 0$ and satisfy one of the following conditions:

1. $q \in Q_0$ and $\phi \in \Psi_n[h, q]$.
2. q is univalent in U and $\phi \in \Psi_n\ h, q_\rho]$, for some $\rho \in (0,1)$.
3. q is univalent in U and there exists $\rho_0 \in (0,1)$ such that $\phi \in \Psi_n\ h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$. If $f \in \mathcal{A}(p)$satisfies (2.11), then

$F_{\lambda,p}^m(f * g)(z) \prec q(z),$
and q is the best dominant.

Proof. By using Theorem 2.2 and Theorem 2.3, we deduce that q is a dominant of (2.11). Since q satisfies (2.12), it is also a solution of (2.11) and therefore q will be dominated by all dominants of (2.11). Hence, q is the best dominant of (2.11).
In the particular case $q(z) = Mz, M > 0$, and in view of the Definition 1.4, the class of admissible function $\Psi_n[\Omega, \mathrm{q}]$ denoted by $\Psi_n[\Omega, \mathrm{M}]$ is described below.

Definition 2.4 Let Ω be a set in $\mathbb{C}$, $M > 0$. The class of admissible functions $\Psi_n[\Omega, \mathrm{M}]$ consists of those functions $\phi :\ \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:

$$\phi\left(Me^{i\theta}, \frac{d + \frac{p(1-\lambda)}{\lambda} Me^{i\theta}}{\frac{p}{\lambda}}, \frac{L + \left[\left(1 + \frac{2p(1-\lambda)}{\lambda}\right) d + \frac{p^2(1-\lambda)^2}{\lambda^2}\right] Me^{i\theta}}{\frac{p^2}{\lambda^2}} ; z \right) \notin \Omega,$$
where $\lambda > 0$, $\theta \in R$, $R(Le^{i\theta}) \geq d(d-1)M$ for all real $\theta, d\ \geq 1, z \in U$.

Corollary 2.2 Let $\phi \in \Psi_n[\Omega, q]$. If $f \in \mathcal{A}(p)$ satisfies

$$\phi\big(F_{\lambda,p}^m(f * g)(z)\ , F_{\lambda,p}^{m+1}(f * g)(z)\ , F_{\lambda,p}^{m+2}(f * g)(z)\ ;\ z\big)\ \in \Omega,$$

where $\lambda > 0$, $m \in N_0$, $z \in U$ *and* $M > 0$. Then

$$\left|F_{\lambda,p}^m(f * g)(z)\right| < M\ , (z \in U).$$

Proof By using Theorem 2.1 gives

$F_{\lambda,p}^m(f * g)(z) \prec q(z) = Mz$
$F_{\lambda,p}^m(f * g)(z) \prec q(z) = Mw(z).$

Hence

$\left|F_{\lambda,p}^m(f * g)(z)\right| < M,$ where $|w(z)| < 1.$

In the special case $\Omega = q(U) =\ \{w :\ |w| < 1\}$ the class $\Psi_n[\Omega, M]$ is simply denote by $\Psi_n[M]$.

Corollary 2.3 Let $\phi \in \Psi_n[\Omega, q]$. If $f \in \mathcal{A}(p)$ satisfies
$\left|\phi(F_{\lambda,p}^m(f * g)(z)\ , F_{\lambda,p}^{m+1}(f * g)(z), F_{\lambda,p}^{m+2}(f * g)(z)\ ; z)\right| < M,$
where $\lambda > 0$, $m \in N_0$, $z \in U$ *and* $M > 0$. Then
$\left|F_{\lambda,p}^m(f * g)(z)\right| < M\ , \qquad (z \in U).$

Differential superordination results associated with linear operator

<u>DIFINITION 3.1</u> Let Ω be a set in $\mathbb{C}$; $q \in Q_0 \cap H[0, p], zq'(z) \neq 0$.The class of admissible function $\Psi'_n[\Omega, \mathrm{q}]$ consists of those function $\phi :\ \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:

$\phi(u, v, w\ ;\ \xi)$ ⃞ Ω,
whenever

$$u = q(z), v = \frac{\frac{1}{k} zq'(z) + \frac{p(1-\lambda)}{\lambda} q(z)}{\frac{p}{\lambda}},$$

and

$$Re\left\{\frac{p^2 w + 2p^2(1-\lambda)v - 3p^2(1-\lambda)^2 u}{\lambda p v - p\lambda(1-\lambda)u}\right\} \leq \frac{1}{k}\, Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\},$$

where $z \in U, \xi \in \partial U \backslash E(q), \lambda > 0$ and $k \geq p.$

Theorem 3.1 Let$\phi \in \Psi_n[\Omega, q]$. If $f \in \mathcal{A}(p)$,$F_{\lambda,p}^m(f * g)(z) \in H_0$ and
$\phi\left(F_{\lambda,p}^m(f * g)(z)\,, F_{\lambda,p}^{m+1}(f * g)(z)\,, F_{\lambda,p}^{m+2}(f * g)(z)\right)$

Is univalent in U, then

$$\Omega \sqsubset \phi\left(F_{\lambda,p}^m(f * g)(z)\,, F_{\lambda,p}^{m+1}(f * g)(z)\,, F_{\lambda,p}^{m+2}(f * g)(z)\right) \tag{3.1}$$

where $\lambda > 0\,, m \in N_0\,, z \in U$, implies
$q(z) \prec F_{\lambda,p}^m(f * g)(z), (z \in U).$

Proof From (2.8) and (3.1), we have

Ω □ $\psi(G(z), zG'(z), z^2G''(z);\ z\,, \xi)$, (z □ U).

From (2.6), we see that the admissibility condition for $\phi \in \Psi'_n[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.5. Hence and by Lemma 1.2 we get $q(z) \prec G(z)$.

$q(z) \prec F_{\lambda,p}^m(f * g)(z), (z \in U).$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathrm{U})$ for some conformed mapping $h(z)$ of U onto Ω. In this case the class $\Psi'_n[h(U), q]$ is written as $\Psi'_n[h, q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.1.

Theorem 3.2 Let $h(z)$ is analytic on U and $\phi \in \Psi'_n[h, q]$. If $f \in \mathcal{A}(p), F_{\lambda,p}^m(f * g)(z) \in H_0$ and

$\phi\left(F_{\lambda,p}^m(f * g)(z)\,, F_{\lambda,p}^{m+1}(f * g)(z)\,, F_{\lambda,p}^{m+2}(f * g)(z)\right)$ is univalent in U, then

$$h(z) \prec \phi\left(F_{\lambda,p}^m(f * g)(z)\,, F_{\lambda,p}^{m+1}(f * g)(z)\,, F_{\lambda,p}^{m+2}(f * g)(z)\right), \tag{3.2}$$

where $\lambda > 0\,, m \in N_0\,, z \in U$, implies

$q(z) \prec F_{\lambda,p}^m(f * g)(z), (z \in U).$

Proof From (3.2), we get

$h(z) = \Omega \subset (F_{\lambda,p}^m(f * g)(z)\,, F_{\lambda,p}^{m+1}(f * g)(z)\,, F_{\lambda,p}^{m+2}(f * g)(z)),$
and also by Theorem 3.1, we get

$q(z) \prec F_{\lambda,p}^m(f * g)(z), (z \in U).$

Theorems 3.1 and 3.2, can only be used to obtain subordinations of differential superordination of the form (3.1) or (3.2). The following Theorem proof the existence of the best subordinate of (3.2) for certain ϕ.

Combining Theorem 2.2 and 3.2, we obtain the following sandwich type Theorem.

Corollary 3.1 Let $h_1(z)$ and $q_1(z)$ be analytic functions in U , $h_2(z)$ be univalent function in $U, q_2(z) \in H_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Psi_n[h_2, q_2] \cap \Psi'_n[h_1, q_1]$. If $f \in \mathcal{A}(p), F_{\lambda,p}^m(f * g)(z), \in H[0, p] \cap H_0$ and
□$\left(F_{\lambda,p}^m(f * g)(z), F_{\lambda,p}^{m+1}(f * g)(z), F_{\lambda,p}^{m+2}(f * g)(z),\right)$, is univalent in U, then

$$h_1(z) \prec \phi\left(F_{\lambda,p}^{m}(f*g)(z), F_{\lambda,p}^{m+1}(f*g)(z), F_{\lambda,p}^{m+2}(f*g)(z),\right) \prec h_2(z) \quad (3.3)$$

when $\lambda > 0$, $m \in N_0$, $z \in U$, implies that

$$q_1(z) \prec F_{\lambda,p}^{m}(f*g)(z), \prec q_2(z), \ (z \in U).$$

References

1. M.K Aouf and A.O. Mostafa, Sandwich theorems for analytic functions defined by convolution. Acta. Univ. Apulensis, No. 21, (2010), pp. 7-20.
2. M.K Aouf and A.O. Mostafa, Subordination results for a class of multivalent non – bazilevic analytic functions defined by linear operator.Acta. Univ. Apulensis, No. 31, (2012), pp. 307 – 320.
3. T. Bulboaca, Differential subordinations and superordinations. Recent Results, House of Scientific Book Publ. Cluj – Napoca, 2005.
4. Cãtas, On certain classes of p-valent functions defined by mutiplier transformations, in Proc. Book of the Internat. Symposium on Geometric Function Theory and Appls. Istanbul. Turkey, (August 2007), pp. 241 – 250.
5. N.E. Cho, Strong differential subordination properties for analytic functions involving the Komatu integral operator. (BVP), (2013), pp. 1-14.
6. J. Dziok and H.M. Srivastava classes of analytic functions associated with the generalized hypergeometric function. Appl. Math. Comput. No. 103, (1999), pp. 1-13.
7. S.S. Miller and P.T. Mocanu, Second order differential inequalities in the complex plan. *J. Math. Anal. Appl*. 65 (1978), pp. 298 – 305.
8. S.S. Miller and P.T. Mocanu, Differential subordinations and univalent functions. *Michigan math. J*. 28 (1981), pp. 157 – 171.
9. S.S. Miller and P.T. Mocanu, Differential subordination. Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York and Basel. 2000.
10. S.S. Miller and P.T. Mocanu, Subordination of differential superordinations, complex varibals Theory Appl. 48, No. 10 (2003), pp.815-826.
11. N.M.Mostafa and M. Darus, Differential subordination and superordiantion of analytic functions defined by a generalization derivative operator, (*IJPAM*), Vol. 79, No. 2, (2012), pp.309-328.
12. G.I. Orosand G.Oros, Strong differential suberordination. *Turk. J. Math*. 33, (2009), pp. 249-257.
13. G.I.Oros, Strong differential suberordination.Acta. Unv. Apulensis, Mat. Inform. 19, (2009), pp. 101-106.
14. C. Selvaraj and K.A. Selvakumaran, On certain classes of multivalent functions involving a generalizatied differential operator. Bull, Korean Math. Soc., 46 (5), (2009), pp. 905-915.

How to cite this article:

Abdul Rahman S. Juma *et al*., On Second – order Differential Subordination and Superordination of Analytic And Multivalent Functions. *International Journal of Recent Scientific Research Vol. 6, Issue, 5, pp.3826-3833, May, 2015*

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