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Parvinder Singh

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RESEARCH ARTICLE

INTRODUCTION TO LEBESGUE INTEGRATION

Parvinder Singh

Department of Mathematics, S.G.G.S. Khalsa College, Mahilpur

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ABSTRACT

In this article, we define the integral of real-valued functions on an arbitrary measure space and derive some of its basic properties. We refer to this integral as the Lebesgue integral, whether or not the domain of the functions is subset of equipped with Lebesgue measure. The Lebesgue integral applies to a much wider class of functions than the Riemann integral and is better behaved with respect to point wise convergence. We carry out the definition in three steps: first for positive simple functions, then for positive measurable functions, and finally for extended real-valued measurable functions and gives the proof of the Fatou's Lemma and at the end proves the Lebasgue Dominated Convergence Theorem.

INTRODUCTION

Definition: Let ϕ be any non zero simple function and $\phi = \sum_{i=1}^{n} \alpha_i C_{E_i}$ where α_i are non zero distinct and E_i are non empty disjoint measurable sets. Then $\phi = \sum_{i=1}^{n} \alpha_i C_{E_i}$ is called the Standard form or Canonical form of ϕ.

Definition: Let ϕ be any non negative simple function, then if ϕ =0 on X then we define $\int_X \phi \, d\mu = 0$.

Suppose $\phi \neq 0$ and $\phi = \sum_{i=1}^{n} \alpha_i C_{E_i}$ is the standard form of ϕ, then we define

$\int_X \phi \, d\mu = \sum_{i=1}^{n} \alpha_i \mu(E_i)$ where E_i are all disjoint measurable sets.

Theorem: Let ϕ and ψ are non negative simple functions, α be any non negative real number. then

(1) $\int \alpha\phi = \alpha \int \phi$ and (2) $\int (\phi + \psi) = \int \phi + \int \psi$

Proof: Let $\alpha_1, \alpha_2, \dots \dots \alpha_n$ are the values of ϕ and $E_i = \{ \phi = \alpha_i \}$ then $\phi = \sum_{i=1}^{n} \alpha_i C_{E_i}$ where α_i are non negative, E_i are disjoint measurable sets and X = $\bigcup_{1=1}^{n} E_i$

Similarly Let $\beta_1, \beta_2, \dots \dots \beta_m$ be all the values of ψ, $F_j = \{ \psi = \beta_j \}$ then $\psi = \sum_{j=1}^{m} \beta_j C_{F_j}$ where β_j are all non negative and F_j are all disjoint and measurable and $\bigcup_{j=1}^{m} F_j$ = X.

(1) If $\alpha = 0$ then both sides are zero, hence equal.

Corresponding author:* **Parvinder Singh
Department of Mathematics, S.G.G.S. Khalsa College, Mahilpur

Let $\alpha > 0$ then $\alpha\,\phi = \sum_{i=1}^{n}(\alpha\alpha_i)C_{E_i} = \sum_{i=1}^{n}(\alpha_i^*)C_{E_i}$ where $\alpha_i^* = \alpha\,\alpha_i$, α_i^* are all non negative, E_i are all disjoint measurable,

Therefore $\int \alpha\phi = \sum_{i=1}^{n}(\alpha_i^*)\mu(E_i) = \sum_{i=1}^{n}(\alpha\alpha_i)\mu(E_i) = \alpha\sum_{i=1}^{n}(\alpha_i)\mu(E_i) = \alpha\int\phi$.

(2) $\phi + \psi = \sum_{i=1}^{n}\sum_{j=1}^{m}(\alpha_i + \beta_j)\ C_{E_i\cap F_j}$ where $\alpha_i + \beta_j$ are non negative, $E_i \cap F_j$

are measurable.

Hence $\int \phi + \psi = \sum_{i=1}^{n}\sum_{j=1}^{m}(\alpha_i + \beta_j)\ \mu(E_i \cap F_j)$

$= \sum_{i=1}^{n}\sum_{j=1}^{m}(\alpha_i)\ \mu(E_i \cap F_j) + \sum_{i=1}^{n}\sum_{j=1}^{m}(\beta_j)\ \mu(E_i \cap F_j)$

$= \sum_{i=1}^{n}(\alpha_i)\sum_{j=1}^{m}\mu(E_i \cap F_j) + \sum_{i=1}^{n}(\beta_j)\sum_{j=1}^{m}\mu(E_i \cap F_j)$

$= \sum_{i=1}^{n}\alpha_i\mu[\bigcup_{j=1}^{m}(E_i \cap F_j)] + \sum_{j=1}^{m}\beta_j\mu[\bigcup_{i=1}^{n}(E_i \cap F_j)]$

$= \sum_{i=1}^{n}\alpha_i\mu[E_i \cap (\bigcup_{j=1}^{m}(F_j)] + \sum_{j=1}^{m}\beta_j\mu[(\bigcup_{i=1}^{n}(E_i) \cap F_j)]$

$= \sum_{i=1}^{n}\alpha_i\mu[E_i \cap \mathrm{X}] + \sum_{j=1}^{m}\beta_j\mu[\mathrm{X} \cap F_j)] = \sum_{i=1}^{n}\alpha_i\mu[E_i] + \sum_{j=1}^{m}\beta_j\mu[F_j] = \int\phi + \int\ \psi$.

Hence $\int \phi + \psi = \int\phi + \int\ \psi$.

Cor(1): If $\phi_1, \phi_2 \dots\dots\dots \phi_n$ are n non negative simple functions then above said result can be extended as $\int(\phi_1 + \phi_2 + \cdots \dots . + \phi_n) = \int\phi_1 + \int\phi_2 + \cdots \dots\dots + \int\phi_n$.

Cor (2): If ϕ & ψ are non negative simple functions and α, β are non negative real numbers then $\int(\alpha\phi + \beta\ \psi) = \alpha\int\phi + \beta\int\psi$.

Cor (3): If $\phi_1, \phi_2 \dots\dots\dots \phi_n$ are n non negative simple functions and $\alpha_1, \alpha_2, \dots\dots\dots, \alpha_n$ are non negative real numbers, then $\int\sum_{i=1}^{n}\alpha_i\phi_i = \sum_{i=1}^{n}\alpha_i\int\phi_i$.

Cor (4): Let $\phi = \sum_{i=1}^{n}(\alpha_i)C_{E_i}$ where α_i are non negative and E_i are measurable then $\int\phi = \sum_{i=1}^{n}\alpha_i\mu(E_i)$.

Definition: Let ϕ be any non negative simple function and E be any measurable set. Let ϕ^* be the restriction of ϕ to E. Then we define $\int_E \phi = \int \phi^*$.

Note: $\int_E \phi = \int \phi C_E$

Theorem: Let ϕ be a non negative simple function. For measurable set E

define $\lambda(\mathrm{E}) = \int_E \phi\ \mathrm{d}\mu$, Then λ is a measure.

Proof: Obviously $\lambda \geq 0$. Moreover $\lambda(\phi) = \int_\phi \phi = \int \phi C_\phi = 0$

But E being measurable set, $\phi = \sum_{i=1}^{n}\alpha_i C_{E_i}$ be the standard form of ϕ. Fof any measurable set A we define $\mu_i(\mathrm{A}) = \mu(\mathrm{A} \cap E_i)$ then μ_i is a measure.

We have $\lambda(E) = \int_E \phi = \int \phi C_E = \int \sum_{i=1}^{n}\alpha_i\, C_{E\cap E_i} = \sum_{i=1}^{n}\alpha_i\,\mu_i(\mathrm{E})$

Hence $\lambda(E) = (\sum_{i=1}^{n}\alpha_i\,\mu_i)(E_i)\ \forall$ measurable set E_i.

This means $\lambda = \sum_{i=1}^{n}\alpha_i\,\mu_i$, Since $\alpha_i \geq 0$ for every $1 \leq i \leq n$. It follows that λ is a measure.

Cor (1): If ϕ is any non negative simple function, A and B are any measurable disjoint sets then $\int_{A\cup B}\phi = \int_A \phi + \int_B \phi$ but $\lambda(E) = \int_E \phi$ then λ is a measure.

Then $\lambda(A \cup \mathrm{B})) = \lambda(A) + \lambda(\mathrm{B}) \Rightarrow \int_{A\cup B}\phi = \int_A \phi + \int_B \phi$.

Cor (2): If N be a null set and ϕ is a non negative simple function.

Then $\int_N \phi = 0 \Rightarrow \lambda(N) = \int_N \phi = 0$.

Cor (3): Let $\phi = 0$ a. e. then $\int \phi = 0$
Suppose $\phi = 0$ on X-N where N is a null set. Then $\int_X \phi = \int_{(X-N)\cup N} \phi$
$= \int_{X-N} \phi + \int_X \phi = 0+0 = 0$.

Definition: Let f be any non negative measurable function, we define
$\int f = \text{Sup}\{ \int \phi \,/\, 0 \le \phi \le f : \phi \text{ is simple}\}$

Definition: Let f be any non negative measurable function and E be any measurable set. Let f^* be the restriction of f to E, Define $\int_E f = \int f^*$.

Note: $\int_E f = \int f C_E$.

Theorem: Let f and g be any non negative measurable functions, then

1. $\int f = 0$ if $f = 0$ a. e.
2. $\int_N f = 0$ if N is a null set.
3. $\int_A f \le \int_B f$ if $A \subset B$
4. $\int f \le \int g$ if $f \le g$ a.e.
5. $\int f = \int g$ if $f = g$ a.e.
6. $\int_{A\cup N} f = \int_A f$ for every null set N.

Proof:

1. Suppose $f = 0$ a.e. Let $0 \le \phi \le f$, ϕ be any simple function. It follows that $\phi = 0$ a.e.
   $\Rightarrow \int \phi = 0 \Rightarrow \text{Sup}\{ \int \phi \,/\, 0 \le \phi \le f : \phi \text{ is simple}\} = 0 \Rightarrow \int f = 0$.
2. Let $f^* = f\, C_N$ then $f^* = 0$ a.e.
   $\Rightarrow \int f^* = 0 \Rightarrow \int_N f\, C_N = 0 \Rightarrow \int_N f = 0$
3. Let $A \subset B$ be any measurable set. Let ϕ be any simple function such that $0 \le \phi \le f_{C_A}$
   Since $A \subset B$, we have $C_A \le C_B \Rightarrow f_{C_A} \le f_{C_B} \Rightarrow 0 \le \phi \le f_{C_B} \Rightarrow \int \phi \le \int f_{C_B}$
   $\Rightarrow \text{Sup}\{\int \phi\} \le \int f_{C_B} \Rightarrow \int f_{C_A} \le \int f_{C_B}$
   $\Rightarrow \int_A f \le \int_B f$.

4. Suppose $f \le g$ on X-N where N is a null set. Let ϕ be a simple function such that
   $0 \le \phi \le f$, Then $0 \le \phi \le f$ on X-N $\Rightarrow 0 \le \phi \le g$ on X-N. $\Rightarrow \int_{X-N} \phi \le \int_{X-N} g$
   $\Rightarrow \int_{X-N} \phi \le \int_X g$ [Because $\int_{X-N} g \le \int_X g$]
   $\Rightarrow \int_{X-N} \phi + \int_N \phi \le \int g$ $[\int_N \phi = 0]$
   $\Rightarrow \int_{X-N} \phi \le \int g \Rightarrow \int \phi \le \int g \Rightarrow Sup \int \phi \le \int g \Rightarrow \int f \le \int g$.
5. Suppose $f = g$ a.e. Then $f \le g$ a.e. $\Rightarrow \int f \le \int g$
   Similarly $g \le f$ a.e. $\Rightarrow \int g \le \int f$ Hence $\Rightarrow \int f = \int g$.
   Let ϕ be any simple function such that $0 \le \phi \le f$ on $A\cup N$.
6. Then $0 \le \phi \le f$ on A $\Rightarrow \int_A \phi \le \int_A f \Rightarrow \int_A \phi + \int_N \phi \le \int_A f$ $[\int_N \phi = 0]$
   $\Rightarrow \int_{A\cup N} \phi \le \int_A f \Rightarrow \text{Sup} \int_{A\cup N} \phi \le \int_A f$ From (3) we get
   $\Rightarrow \int_A f \le \int_{A\cup N} f$ Hence $\int_{A\cup N} f = \int_A f$.

Monotone Convergence Theorem: Let $\{f_n\}$ be an increasing sequence of non negative measurable functions and f be non negative such that $f_n \to f$ a.e. then $\int f_n \to \int f$.

Proof: First we note that f is measurable because the limit of a sequence of measurable functions is measurable.

And $f_n \le f$ $\forall$ n [Because for increasing sequence Limit $A_n = \bigcup_1^\infty A_n$

And for decreasing sequence Limit $A_n = \bigcap_1^\infty A_n$]

$\Rightarrow \int f_n \le \int f \;\forall n \Rightarrow \text{Limt} \int f_n \le \int f$ ……………(1)

Let $0 < \alpha < 1$ be any real number. And $0 \le \phi \le f$ be any simple function.

Define $A_n = \{\alpha\phi \le f_n\}$, Since $\alpha\phi$, f_n are measurable functions, it follows that A_n is measurable for all n. As (f_n) is an increasing sequence, it is clear that A_n is also an increasing sequence. Let $x \in X$, Suppose $f(x) = 0$ then $\phi(x) = 0 \Rightarrow \alpha\,\phi(x) = 0$

$\Rightarrow \alpha\,\phi(x) \le f_n(x) \ \ \forall\, n \Rightarrow x \in \{\alpha\,\phi \le f_n\} \Rightarrow x \in A_n \Rightarrow x \in \bigcup_1^\infty A_n$

But $f(x) > 0$ then $\alpha\,\phi(x) \le \alpha\, f(x) < f(x) \Rightarrow \alpha\,\phi(x) < f(x)$. As $f_n \to f$ we can find k s.t. $f_k(x) > \alpha\,\phi(x) \Rightarrow x \in \{\alpha\,\phi \le f_k\} \Rightarrow x \in A_k,\ x \in \bigcup_1^\infty A_n$ shows that $x \in \bigcup_1^\infty A_n$,

Hence $\bigcup_1^\infty A_n = X$ ………………. (2)

For measurable set E define $\lambda(E) = \int_E \phi$, then λ is a measure

From (2) we have $(A_n) \uparrow X$ therefore $\lambda(A_n) \uparrow \lambda(X)$ [Because of the continuity of the measure for increasing limits]

$\Rightarrow$ Limit $\lambda(A_n) = \lambda(X) \Rightarrow \text{Limt}\int_{A_n} \phi = \int \phi$ ……………….. (3)

By definition of A_n we have $\alpha\phi \le f_n$ on $A_n \Rightarrow \int_{A_n} \alpha\phi \le \int_{A_n} f_n \Rightarrow \alpha\int_{A_n}\phi \le \int f_n$ $[\int_{A_n} f_n \le \int_X f_n]$

$\Rightarrow$ Limt $\alpha\int_{A_n}\phi \le \text{Limt}\int f_n \Rightarrow \alpha\ \text{Limt}\int_{A_n}\phi \le \text{Limt}\int f_n \Rightarrow \alpha\int\phi \le \text{Limt}\int f_n$ [From (3)]

Taking limit as $\alpha \to 1$ we obtain $\int\phi \le \text{Limt}\int f_n \Rightarrow Sup\{\int\phi\} \le \text{Limt}\int f_n$

$\Rightarrow \int f \le \text{Limt}\int f_n$ ………….. (4)

From (1) and (4) we get $\text{Limt}\int f_n = \int f$ proved.

Note: If the sequence $\{f_n\}$ is not an increasing sequence then Monotone Convergence Theorem does not hold. Consider the following example.

Let $(\mathcal{R}, \mathcal{M}, m)$ be the L'- measure space. Let $f_n = \frac{1}{n}C_{[0,n]}$ then $f_n \ge 0$, f_n is measurable $\forall$ n and $f_n \to 0$ but $\int f_n \nrightarrow \int 0$

Also Another example let $f_n = \frac{1}{n}C_{[n,\infty]}$ then $f_n \ge 0$ and (f_n) is monotonic decreasing sequence of measurable functions and $f_n \to 0$ uniformly but $\int f_n \nrightarrow \int 0$.

Fatou's Lemma: Let $\{f_n\}$ be a sequence of non negative measurable functions and $f_n \to f$ a.e. on a set E then $\int_E f \le \underline{\text{Limt}}\int_E f_n$.

Proof: Without any loss of generality we may assume that the convergence being everywhere. Since integrals over set of measure zero are zero. Let h be a bounded measurable function which is not greater than f and which vanish outside a set E' of finite measure.

Define h_n by letting $h_n(x) = \min\{h(x), f_n(x)\}$ then h_n is bounded by the bound of h and vanish outside E'.

Now $h_n \to h$ for each $x \in E'$ then we have $\int_E h = \int_{E'} h = \text{limt}\int_{E'} h_n \le \underline{\text{Limt}}\int_{E'} f_n$.

Taking sup over h we get $\int_E h \le \underline{\text{Limt}}\int_E f_n$. Proved.

Another Proof: Let $f = \underline{\text{Limt}}\, f_n$. Define $g_n = \text{Inf}\{f_n, f_{n+1}, f_{n+2}, \dots\dots\}$ $= \text{Inf}\{f_k / k \ge n\}$ Then $\{g_n\}$ is a monotonic increasing sequence of non negative measurable functions. Then Limit $(g_n) = \sup_n(g_n) = \sup_n\{\text{Inf}(f_k)/k \ge n\} = \underline{\text{Limt}}\, f_n = f$.

Thus From Monotone Convergence Theorem $\int f = \text{limit}\int g_n$ …………… (1)

$g_n \le f_k\ \forall\, k \ge n \Rightarrow \int g_n \le \int f_k \ \ \forall\, k \ge n \Rightarrow \int g_n \le \underline{\text{Limt}}\int f_k \Rightarrow \text{Limit}\int g_n \le \underline{\text{Limt}}\int f_k$

$\Rightarrow \int f \le \underline{\text{Limt}}\int f_k$ [From (1)]

$\Rightarrow \int \underline{\text{Limt}}\,(f_n) \le \underline{\text{Limt}}\int f_n$. Proved.

Cor: If $\{f_n\}$ be any sequence of non negative measurable functions and $f_n \to f$ then $\Rightarrow \int f \le \underline{\text{Limt}}\int f_n$

Preposition: Let f & g are non negative measurable functions and α be any non negative constant, then

1. $\int \alpha f = \alpha \int f$ and (2) $\int (f+g) = \int f + \int g$

Proof: Since f & g are non negative measurable functions, we can find increasing sequences $\{\phi_n\}$ and $\{\psi_n\}$ of non negative simple functions s.t. $\phi_n \to f$ and $\psi_n \to g$.

1. $(\alpha\phi_n)$ is an increasing sequence of non negative simple functions and $\alpha\phi_n \to \alpha f$ therefore $\int \alpha\phi_n \to \int \alpha f$ [By M.C.T.]
   $\Rightarrow \int \alpha f$ = limt $\int \alpha\phi_n$ = limt $\alpha \int \phi_n$ = α limt $\int \phi_n = \alpha \int f$ [By M.C.T.]
2. $(\phi_n + \psi_n)$ is monotone increasing sequence of non negative simple functions and
   $\phi_n + \psi_n \to f+g$

Therefore $\int (\phi_n + \psi_n) \to \int (f+g)$ [By M.C.T.]
$\int (f+g)$ = limt $\int (\phi_n + \psi_n)$ = limit ($\int (\phi_n)$ +$\int (\psi_n)$)
= limt ($\int (\phi_n)$ + limt $\int (\psi_n)$ = $\int f + \int g$ [By M.C.T.]
Proved.

Cor (1) Let f & g are non negative measurable functions and α *and* β be any non negative constants, then
$\int (\alpha f + \beta g) = \alpha \int f + \beta \int g$.

Cor (2) If $f_1, f_2, \ldots, f_n$ are non negative measurable functions and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are non negative constants then $\int \sum_{i=1}^{n} \alpha_i f_i$ = $\sum_{i=1}^{n} \alpha_i \int f_i$.

Theorem: Let $\sum_{n=1}^{\infty} f_n$ be a series of non negative measurable functions,

then $\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n$.

Proof: Let $g_n = \sum_{i=1}^{n} f_i$ and $f = \sum_{n=1}^{\infty} f_n$ then $(g_n) \to f$ and (g_n) is an increasing sequence of non negative measurable functions. Hence by M.C.T. we have

$\int f$ = limit $\int g_n \Rightarrow \int \sum_{n=1}^{\infty} f_n$ = limt $\int (f_1 + f_2 + \cdots + f_n)$ =Limt ($\int f_1 + \int f_2 + \cdots + f_n$)
= limit ($\sum_{i=1}^{n} \int f_i$) = $\sum_{i=1}^{\infty} \int f_i$ which shows that $\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n$.

Preposition: Let f be non negative measurable function. For any measurable set E define $\lambda(E) = \int_E f$, then λ is a measure.
Proof: It is clear that $\lambda \geq 0$ and $\lambda(\phi) = 0$

Let (E_n) be any sequence of disjoint measurable sets and E = $\bigcup_{n=1}^{\infty} E_n$. since the sets are disjoint we get

$$C_E = \sum_{n=1}^{\infty} C_{E_n} \Rightarrow \int C_E = \sum_{n=1}^{\infty} \int C_{E_n} \Rightarrow \int f C_E = \int \sum_{n=1}^{\infty} f C_{E_n} = \sum_{n=1}^{\infty} \int f C_{E_n}$$

$\Rightarrow \int_E f = \sum_{n=1}^{\infty} \int_{E_n} f \Rightarrow \lambda(E) = \sum_{n=1}^{\infty} \lambda(E_n)$. Which proves that λ is a measure.

Cor(1): If A and B are disjoint measurable sets, then $\int_{A \cup B} f = \int_A f + \int_B f$.

Cor(2): Let (E_n) is any sequence of disjoint measurable sets, then $\int_{\bigcup_1^{\infty} E} f = \sum_1^{\infty} \int_{E_n} f$.

Theorem: Let f be non negative measurable function, then $\int f = 0$ iff $f = 0$ a.e.

Proof: If $f = 0$ a.e. then $\int f = 0$ obviously.
Conversely let $\int f = 0$ a.e., to show that $f = 0$ a.e.
Define $E_n = \{ f > \frac{1}{n} \}$, and E = $\{ f > 0 \}$ then E = $\bigcup_1^{\infty} E_n$.
Since $\frac{1}{n} C_{E_n} \leq f$ we get $\int \frac{1}{n} C_{E_n} \leq \int f \Rightarrow \frac{1}{n}\mu(E_n) \leq 0$ $\forall$ n $\Rightarrow \mu(E_n) \leq 0$ $\forall$ n $\Rightarrow \mu(E_n) = 0$ $\forall$ n
$\Rightarrow \sum_1^{\infty} \mu(E_n) = 0 \Rightarrow \mu$ (E) $\leq \sum_1^{\infty} \mu(E_n)$ We find that μ (E) = 0, Thus E is a null set and $f = 0$ on X-E. Proves that $f = 0$ a.e.

Definition: Let f be any non negative measurable function. If $\int f < \infty$ then f is said to be integrable.

Note: $f \vee g = \frac{(f+g)+|f-g|}{2}$ and $f \wedge g = \frac{(f+g)-|f-g|}{2}$

Also $f^+ = f \vee 0 = \frac{(f)+|f|}{2}$ and $f^- = -\{f \wedge 0\} = -\frac{(f)-|f|}{2} = \frac{(|f|)-f}{2} \Rightarrow f^+ - f^- = f$ and $f^+ + f^- = |f|$

Definition: Let f be any measurable function. If f^+ and f^- both are integrable then f is said to be integrable and we define $\int f = \int f^+ - \int f^-$.

Theorem: Let f be measurable function then (1) f is integrable iff $|f|$ is integrable and $|\int f| \leq \int |f|$. (2) If g is integrable and $|f| \leq g$ then f is integrable.

Proof: (1) Suppose $|f|$ is integrable. Then $f^+ + f^- = |f|$, we see that $f^+ \leq |f|, f^- \leq |f|$
$\Rightarrow \int f^+ \leq \int |f|, \int f^- \leq \int |f| \Rightarrow \int f^+ < \infty$ *and* $\int f^- < \infty \Rightarrow f^+$ and f^- both are integrable.
$\Rightarrow f$ is integrable.

For the converse assume that f is integrable. That means f^+ and f^- both are integrable.
$\Rightarrow \int f^+ < \infty$ *and* $\int f^- < \infty$ But $|f| = f^+ + f^-$ which gives $\int |f| = \int (f^+ + f^-)$
$= \int f^+ + \int f^- < \infty \Rightarrow \int |f| < \infty$ shows that $|f|$ is integrable.
Further $|\int f| = |\int f^+ + \int f^-| \leq |\int f^+| + |\int f^-| = \int f^+ + \int f^- = \int (f^+ + f^-) = \int |f|$.
(1) Let g be integrable and $|f| \leq g$ *then* $\int |f| \leq \int g < \infty \Rightarrow |f|$ is integrable
(2) $\Rightarrow f$ is integrable. [From part (1)]

Definition: Let f be any measurable function and E be any measurable set. Let $f^* = f/E$. If f^* is integrable then we say that f is integrable over E and $\int_E f = \int_E f^*$.

Remark: f is integrable over E iff f_{c_E} is integrable and $\int_E f = \int f_{c_E}$.

Theorem: Let f be any measurable function, then

1. If f is integrable and E is any measrurable set, then f is integrable over E.
2. $\int_N f = 0$ if N be a null set.
3. $\int f = 0$ if $f = 0$ a.e.
4. If f is integrable and $g = f$ a.e. then g is integrable. And $\int f = \int g$
5. If f is integrable and for a measurable set E $v(E) = \int_E f$ then v is finite, $v(\phi) = 0$ and v is countably additive.

Proof: Suppose f is integrable and E be a measurable set then f^+ and f^- are integrable

$\Rightarrow \int f^+ < \infty$ *and* $\int f^- < \infty$ from $\int_E f^+ \leq \int_X f^+ < \infty$, $\int_E f^- \leq \int_X f^- < \infty$
Note that f^+ and f^- are integrable over E $\Rightarrow f$ is integrable over E.
(2) Let N be a null set then $\int_N f^+ = 0$ and $\int_N f^- = 0 \Rightarrow \int_N f = \int_N f^+ - \int_N f^- = 0 - 0 = 0$
(3) Let $f = 0$ a.e. then $f^+ = 0$ a.e. and $f^- = 0$ a.e. $\Rightarrow \int f^+ = 0$ and $\int f^- = 0$
$\Rightarrow \int f = \int f^+ - \int f^- = 0 \Rightarrow \int f = 0.$
(4) Let f be integrable and $g = f$ a.e then $g^+ = f^+$ a.e. and $g^- = f^-$ a.e.
$\Rightarrow g^+$and g^- are integrable, $\Rightarrow g$ is integrable.
(5) $\int_E f$ is finite for every E $\Rightarrow v(E)$ is finite and $v(\phi) = \int_\phi f = 0$
Let (E_n) be any sequence of disjoint measurable sets then
$v(\bigcup_1^\infty E_n) = \int_{\bigcup_1^\infty E_n} f = \int_{\bigcup_1^\infty E_n} f^+ - \int_{\bigcup_1^\infty E_n} f^- = \sum_1^\infty \int_{E_n} f^+ - \sum_1^\infty \int_{E_n} f^- = \sum_1^\infty (\int_{E_n} f^+ - \int_{E_n} f^-)$
$= \sum_1^\infty \int_{E_n} f = \sum_1^\infty v(E_n).$

Note: (1) $(\alpha f)^+ = \alpha f^+$, $(\alpha f)^- = \alpha f^-$ if $\alpha \geq 0$
and $(\alpha f)^+ = (-\alpha) f^-$, $(\alpha f)^- = (-\alpha) f^+$ if $\alpha < 0$.

Theorem: Let f and g be two integrable functions and α, β be any constants then

1. αf is integrable and $\int \alpha f = \alpha \int f$
2. $f + g$ is integrable and $\int (f + g) = \int f + \int g$.

Proof: (1) If $\alpha = 0$ then $\alpha f = 0, \int \alpha f = 0, \alpha \int f = 0$ Hence no further argument is needed.
Let $\alpha \geq 0$ then $\int (\alpha f)^+ = \int \alpha f^+ = \alpha \int f^+ < \infty$ shows that $(\alpha f)^+$ is integrable.
Also $\int (\alpha f)^- = \int \alpha f^- = \alpha \int f^- < \infty$ shows that $(\alpha f)^-$ is integrable.
And $\int \alpha f = \int (\alpha f)^+ - \int (\alpha f)^- = \alpha \int f^+ - \alpha \int f^- = \alpha[\int f^+ - \int f^-] = \alpha \int f$.
Suppose $\alpha < 0$ then $\int (\alpha f)^+ = \int (-\alpha) f^- = (-\alpha) \int f^- < \infty$
$\Rightarrow \int (\alpha f)^- = \int (-\alpha) f^+ = -\alpha \int f^+ < \infty$ shows that αf is integrable.
$\int \alpha f = \int (\alpha f)^+ - \int (\alpha f)^- = (-\alpha)\int f^- - (-\alpha)\int f^+ = \alpha[\int f^+ - \int f^-] = \alpha \int f$.
(2) Write h $= f + g$ then $h^+ - h^- = f^+ - f^- + g^+ - g^- \Rightarrow h^+ + f^- + g^- = f^+ + g^+ + h^-$

These all are non negative measurable functions

Consequently $\int (h^+ + f^- + g^-) = \int (f^+ + g^+ + h^-)$
Gives $\int h^+ + \int f^- + \int g^- = \int f^+ + \int g^+ + \int h^-$
$\Rightarrow \int h^+ - \int h^- = \int f^+ - \int f^- + \int g^+ - \int g^-$
$\Rightarrow \int h = \int f + \int g \Rightarrow \int (f + g) = \int f + \int g$.

Cor (1) Let f and g be two integrable functions and α, β be any constants then $\alpha f + \beta g$ is integrable and $\int (\alpha f + \beta g) = \alpha \int f + \beta \int g$.
Cor (2) Let $f_1, f_2, \dots\dots, f_n$ be finitely many integrable functions and $\alpha_1, \alpha_2, \dots\dots, \alpha_n$ are constants then then $\alpha_1 f_1 + \alpha_2 f_2 + \cdots \dots + \alpha_n f_n$ is integrable and $\int \sum_{i=1}^{n} \alpha_i f_i = \sum_{i=1}^{n} \alpha_i \int f_i$.

Lebesgue Dominated Convergence Theorem: Let $\{f_n\}$ be any sequence off measurable functions, g be any integrable function such that $|f_n| \leq g$ for all n, If $f_n \to f$ a.e. then f is integrable and $\int f_n \to \int f$.

Proof: Since g is integrable, f_n is measurable and $|f_n| \leq g$ for all n, it follows that f_n is integrable for all n. From $|f_n| \leq g$ we get $-g \leq f_n \leq g$. The fact that $f_n \to f$
gives $-g \leq f_n \leq g \Rightarrow |f| \leq g$.

Thus as $f_n \to f \Rightarrow f = limt\,(f_n)$ and f_n is measurable for all n, It follows that f is measurable. Hence f is integrable.
Now for every n we have $-g \leq f_n \leq g$ ……………….(*)

From (*) we see that $(g + f_n)$ is a sequence of non negative measurable functions. Using Fatou's Lemma, we obtain $\int \underline{\text{Limt}}(g + f_n) \leq \underline{\text{Limt}} \int (g + f_n)$
$\Rightarrow \int (g + f) \leq \underline{\text{Limt}}(\int g + \int f_n) = \int g + \underline{\text{Limt}} \int f_n$
$\Rightarrow \int g + \int f \leq \int g + \underline{\text{Limt}} \int f_n \Rightarrow \int f \leq \underline{\text{Limt}} \int f_n$ …………… (1)

Also from (*) we note that $(g - f_n)$ is a sequence of non negative measurable functions. Using Fatou's Lemma we obtain $\int \underline{\text{Limt}}(g - f_n) \leq \underline{\text{Limt}} \int (g - f_n)$
$\Rightarrow \int (g - f) \leq \underline{\text{Limt}}(\int g - \int f_n) \Rightarrow \int g - \int f \leq \int g - \overline{\text{Limt}} \int f_n \Rightarrow -\int f \leq - \overline{\text{Limt}} \int f_n$
$\Rightarrow \overline{\text{Limt}} \int f_n \leq \int f$ …………… (2)

From (1) and (2) we get $\overline{\text{Limt}} \int f_n \leq \int f \leq \underline{\text{Limt}} \int f_n$
Since $\underline{\text{Limt}} \int f_n \leq \overline{\text{Limt}} \int f_n$, we must have $\underline{\text{Limt}} \int f_n = \overline{\text{Limt}} \int f_n = \int f$
$\Rightarrow$ Limt $\int f_n = \int f$ i.e. $\int f_n \to \int f$. Proved.

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