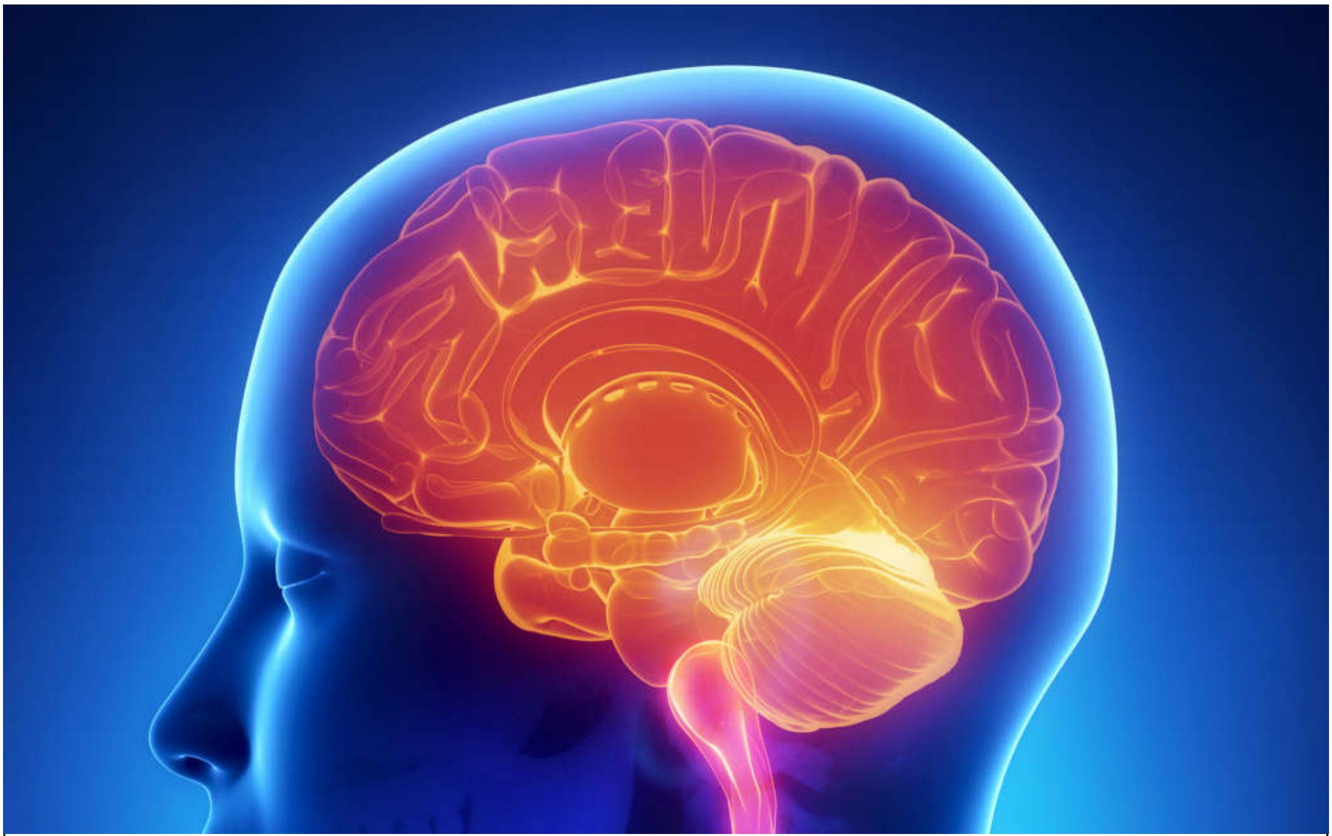

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RESEARCH ARTICLE

COMMON FIXED POINT THEOREMS IN FUZZY METRIC SPACES

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ABSTRACT

In this paper we introduce the concept of semi compatible mappings in the context of a fuzzy metric space and prove results on common fixed point of four self mappings of semi compatible mappings. Our result generalize the result of Singh B., Jain S., and Jain S. [5].

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INTRODUCTION

The concept of fuzzy metric space which was introduced by Kramosil and Michalek (3). Chang (2) obtained a coincidence theorem for fuzzy mappings on topological spaces. Upadhyay and Choudhary (6) also proved a unique fixed point theorem for a self mapping and common fixed point theorem for four self mappings in the context of fuzzy metric spaces. In Cho, Sharma and Sahu (5) introduced the concept of semi compatibility of maps d-complete metric spaces.

Singh and Chauhan (4) prove the existence of unique common fixed point of four self compatible mappings. Here we assume only the semi compatibility of the (A, S) and (B, T) and take only one of the four maps to be continuous and proved a similar theorem on existence of unique common fixed point of the four maps in a fuzzy metric space.

We have used the following definition and notions:

Definition 1 Let X be any set. A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition 2 A binary operation *: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t-norm if, ([0, 1], *) is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all a, b, c, d in [0, 1].

For an example: a * b = ab, a * b = min {a, b}.

Definition 3 The triplet (X, M, *) is called a fuzzy metric space (shortly, a FM-space) if, X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X*X \times [0, 1)$ satisfying the following conditions: for all x, y, z in X, and s, t > 0,

(i) $M(x, y, 0) = 0$, $M(x, y, t) > 0$;
(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(v) $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous.

In this case, M is called a fuzzy metric on X and the function $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t.

Also, we consider the following condition in the fuzzy metric space (X, M, *):

(vi) $\lim_{t \rightarrow \infty} M(x, y, t) = 1$, for all $x, y \in X$.

It is important to note that every metric space (X, d) induces a fuzzy metric space (X,M, *) where $a * b = \min\{a, b\}$ and for all $a, b \in X$, we have $M(x, y, t) = \frac{t}{t+d(x,y)}$, for all t > 0, and $M(x, y, 0) = 0$, so-called the fuzzy metric space induced by the

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metric d.

Definition 4 A sequence $\{x_n\}$ in a fuzzy metric space (X, M, *) is called a Cauchy sequence if, $\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1$ for every t > 0 and for each p > 0.

A fuzzy metric space (X, M, *) is complete if, every Cauchy sequence in X converges in X.

Definition 5 A sequence $\{x_n\}$ in a fuzzy metric space (X, M, *) is said to be convergent to x in X if, $\lim_{n \to \infty} M(x_n, x, t) = 1$, for each t > 0.

Definition 6 A sequence $\{xn\}$ in a fuzzy metric space (X, M, *) is called a Cauchy sequence if, $\lim_{n \to \infty} M(xn+p, xn, t) = 1$ for every t > 0 and for each p > 0.

Definition 7 A fuzzy metric space (X, M, ∗) is said to be complete if every Cauchy sequence is convergent.

Definition 8 A map f :X→X is said to be continuous on x if $f(x_n) \to f(x)$ whenever $x_n \to x$.

Definition 9 Two self mappings A and B of a fuzzy metric space (X, M, *) are said to be compatible if, $\lim_{n \to \infty} M(ABxn, BAxn, t) = 1$ whenever $\{xn\}$ is a sequence such that $\lim_{n\to\infty} Axn = \lim_{n \to \infty} Bxn = x$, for some p in X.

Definition 10 Two self mappings A and B of a fuzzy metric space (X, M, *) are said to be semi compatible if, $\lim_{n \to \infty} M(ABxn, Bxn, t) = 1$ whenever $\{xn\}$ is a sequence such that $\lim_{n \to \infty} Axn = \lim_{n \to \infty} Bxn = x$, for some p in X.

It follows that (A,S) is semi compatible and Ay = Sy imply ASy = SAy by taking {xn} = y and x = Ay = Sy.

Proposition 11 In a fuzzy metric space (X, M, *) limit of a sequence is unique.

Lemma 12 Let (X, M, *) be a fuzzy metric space. If there exists k £ (0, 1) such that $M(x, y, kt) \geq M(x, y, t)$ then $x = y$.

Proposition 13 (A, S) is a semi compatible pair of self maps of a fuzzy metric space (X, M, *) and S is continuous then (A, S)is compatible.

Proof: Consider a sequence {xn} in X such that {Axn} → x and {Sxn} → x. By semi compatibility of (A, S) we have $\lim_{n\to\infty} ASxn = Sx$. As S is continuous we get $\lim_{n\to\infty} SAxn = Sx$.

Now,

$\text{Lim}_{n\to\infty} M(SAxn, ASxn, t) = M(Sx, Sx, t) = 1$.

Hence (A, S) is compatible.

Example Let = [0,2], define

$$Sx=\begin{cases} 1 , & x\in[0,1) \\ 2 , & x=1 \\ (x+3)/5 , & x\in(1,2] \end{cases} \qquad Ax=\begin{cases} 2 , & x\in[0,1) \\ x/2 , & x\in(1,2] \end{cases}$$

and xn = 2 – 1/(2n) and M(x, y, t) = t/[t+|x-y|].

Solution: we have S(1) = A(1) = 2 and S(2) = A(2) = 1. Also SA(1) = AS(1) =1 and SA(2) = AS(2) = 2. Hence Axn → 1 and Sxn → 1, ASxn → 2 and SAxn → 1.

Now,

$\lim_{n\to\infty} M(ASxn, Sy, t) = (2, 2, t) = 1$

and $\lim_{n\to\infty} M(ASxn, SAxn, t) = M(2, 1, t) = t/[t+1] < 1$.
Hence (A, S) is semi compatible but not compatible.

We have to prove that following theorem

Theorem 1 Let (X, M, *) be a complete fuzzy metric space and with continuous t – norm defined by a *b = min{a,b}, for all a, b ∈ [0, 1]. Let A, B, S and T be a mappings from X into itself such that

(1.1) $A(X) \leq T(X)$, $B(X) \leq S(X)$.
(1.2) One of A, B, S and T is continuous.
(1.3) (A, S) and (B, T) are semi compatible pairs of mappings,
(1.4) $M(Ax, By,t) \geq \phi (M(Ax, Sx ,t), M(By,Ty, t), M(Sx, Ty, t), M(Ax, Ty, \alpha t), M(Sx, By, (2-\alpha)t))$

Where $\phi:[0, 1] \to [0,1]$ is continuous function such that $\phi(t) > t$ for some 0 < t < 1and for all and x, y ∈ X, α ∈ (0, 2) and t >0. Then A, B, S and T have a unique common fixed point.

Proof

Let $x_0 \in X$ be an arbitrary point. Then, since $A(X) \leq T(X)$, $B(X) \leq S(X)$, there exists $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Inductively, we construct the sequences $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, for n = 0, 1, 2, ...

Now, we put α = 1 − q with q ∈ (0, 1) in (1.4), then we have

$$M(y_{2n}, y_{2n+1}, t) = M(Ax_{2n}, Bx_{2n+1}, t) \geq \phi(\min\{M(Ax_{2n}, Sx_{2n}, t),$$

$$M(Bx_{2n+1}, Tx_{2n+1}, t), M(Sx_{2n}, Tx_{2n+1}, t), M(Ax_{2n}, Tx_{2n+1}, (1 - q)t), M(Sx_{2n}, Bx_{2n+1}, (1 + q)t)\}).$$

That is,

$$M(y_{2n}, y_{2n+1}, t) \geq \phi(\min\{M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n-1}, y_{2n+1}, (1 + q)t)\})$$
$$\geq \phi(\min\{M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n-1}, y_{2n}, t),$$

$$M(y_{2n-1}, y_{2n+1}, qt)\})$$
$$\geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t) * M(y_{2n}, y_{2n+1}, qt).$$

Since t-norm * is continuous, letting q → 1, we have

$$M(y_{2n}, y_{2n+1}, t) \geq \phi(\min\{M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t)\}$$
$$\geq \phi(\min\{M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t)\}).$$

It follows that, $M(y_{2n}, y_{2n+1}, t) > M(y_{2n-1}, y_{2n}, t)$, since $\phi(t) > t$ for each 0 < t < 1.
Similarly, $M(y_{2n+1}, y_{2n+2}, t) > M(y_{2n}, y_{2n+1}, t)$. Therefore, in general, we have

$$M(y_n, y_{n+1}, t) \geq \phi(M(y_{n-1}, y_n, t)) > M(y_{n-1}, y_n, t).$$

Therefore, $\{M(y_n, y_{n+1}, t)\}$ is an increasing sequence of positive real numbers in [0, 1] and tends to a limit, say $\lambda \leq 1$. We claim that $\lambda = 1$. If $\lambda < 1$, then $M(y_n, y_{n+1}, t) \geq r(M(y_{n-1}, y_n, t))$.

So, on letting $n \to \infty$, we get $\lim_{n\to\infty} M(y_n, y_{n+1}, t) \geq r(\lim_{n\to\infty} M(yn, yn+1, t))$, that is, $\lambda \geq \phi(\lambda) > \lambda$, a contradiction. Thus, we have $\lambda = 1$.

Now, for any positive integer p, we have

$$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t)*M(y_{n+1}, y_{n+2}, t/p) * ... *M(y_{n+p-1}, y_{n+p}, t/p).$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} M(y_n, y_{n+p}, t) \geq 1 *1 * ... * 1 = 1$. Thus, we have $\lim_{n\to\infty} M(y_n, y_{n+p}, t) = 1$. Hence, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete metric space, so the sequence $\{y_n\}$ converges to a point u (say) in X and Hence

$$\{Ax_{2n}\} \to u, \quad \{Sx_{2n}\} \to u \quad (1)$$
$$\{Tx_{2n+1}\} \to u, \ Bx_{2n+1}\} \to u. \quad (2)$$

We first consider the case when S is continuous
In this case

$$SAx_{2n} \to Su, \text{ and } S^2 x_{2n} \to Su. \quad (3)$$

Also as (A, S) is semi compatible

$$ASx_{2n} \to Su. \quad (4)$$

Step I. Take $x = Sx_{2n}$, $y = x_{2n+1}$ in (1.4) we get

$$M(ASx_{2n}, Bx_{2n+1},t) \geq \phi(M(ASx_{2n}, SSx_{2n},t), M(Bx_{2n+1},Tx_{2n+1}, t), M(SSx_{2n}, Tx_{2n+1}, t),$$
$$M(ASx_{2n}, Tx_{2n+1}, \alpha t), M(SSx_{2n}, Bx_{2n+1}, (2-\alpha)t))$$

Taking limit $n \to \infty$ and using equations (1) to (4) we get

$$M(Su, u, t) \geq \phi(M(Su, Su,t), M(u, u, t), M(Su, u, t), M(Su, u, \alpha t), M(Su, u, (2-\alpha)t))$$
$$\geq M(Su, u, t)$$

Hence $\qquad Su=u \qquad (5)$

Step II. Take $x = u$, $y = x_{2n+1}$ in (1.4) we get

$$M(Au, Bx_{2n+1},t) \geq \phi(M(Au, Su,t), M(Bx_{2n+1},Tx_{2n+1}, t), M(Su, Tx_{2n+1}, t),$$
$$M(Au, Tx_{2n+1}, \alpha t), M(Su, Bx_{2n+1}, (2-\alpha)t))$$

Taking limit $n \to \infty$ and using equations (1) to (5) we get

$$M(Au, u, t) \geq \phi(M(Au, u,t), M(u, u, t), M(u, u, t), M(u, u, \alpha t), M(u, u, (2-\alpha)t))$$
$$\geq M(Au, u, t)$$

Hence $\qquad Au = u$
Thus

$$Su = u = Au \quad (6)$$

Step III. As $A(X) \leq T(X)$, $\exists\, z \in X$ such that
$Au = Tz = u$
Put $x = x_{2n}$, $y = z$ in (1.4) we get

$$M(Ax_{2n}, Bz,t) \geq \phi(M(Ax_{2n}, Sx_{2n},t), M(Bz,Tz, t), M(Sx_{2n}, Tz, t), M(Ax_{2n}, Tz, \alpha t), M(Sx_{2n}, Bz, (2-\alpha)t))$$

Taking limit $n \to \infty$ and using equations (1) to (4) we get

$$M(u, Bz, t) \geq \phi(M(u, u, t), M(Bz, u, t), M(u, u, t), M(u, u, \alpha t), M(u, Bz, (2-\alpha)t))$$
$$= M(Bz, u, t)$$

Hence $Bz = u$ and we get $Tz = Bz = u$ and as (B, T) is semi compatible we get $BTz = TBz$. i.e. $Bu = Tu$.

Step IV. Take $x = u$, $y = u$ in (1.4) we get

$$M(Au, Bu,t) \geq \phi(M(Au, Su,t), M(Bu,Tu, t), M(Su, Tu, t), M(Au, Tu, \alpha t),$$
$$M(Su, Bu, (2-\alpha)t))$$

As $Au = Su = u$ and $Bu = Tu$ we get

$$M(u, Tu,t) \geq \phi(M(u, u,t), M(Tu,Tu, t), M(u, Tu, t), M(u, Tu, \alpha t),$$
$$M(u, Tu, (2-\alpha)t))$$
$$\geq M(u, Tu, t)$$

Thus $u = Tu$ and we get $Au = Bu = Su = Tu = u$.

Hence u is a common fixed point A, B, S and T.

Uniqueness

The uniqueness of a common fixed point of the mappings A, B, S and T be easily verified by using (1.4). In fact, if u0 be another fixed point for mappings A, B, S and T. Then, for $\alpha = 1$, we have

$$M(u, u0, t) = M(Au, Bu0, t) \geq \phi(\min\{M(Su, Tu0, t), M(Au, Tu0, t), M(Su,Bu0, t)\}),$$
$$\geq \phi(M(u, u0, t)) > M(u, u0, t),$$ and hence, we get $u = u0$.

This completes the proof of the theorem.
Similarly, we can prove the result using continuity of A or B or T.

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