International Journal Of

Recent Scientific Research

ISSN: 0976-3031

Volume: 7(5) May -2016

FORBIDDEN SUBGRAPHS FOR VICT GRAPH OF A GRAPH

Muddebihal M.H and Jayashree. B. Shetty

THE OFFICIAL PUBLICATION OF
INTERNATIONAL JOURNAL OF RECENT SCIENTIFIC RESEARCH (IJRSR)
http://www.recentscientific.com/ recentscientific@gmail.com

Available Online at http://www.recentscientific.com

International Journal of Recent Scientific Research
Vol. 7, Issue, 5, pp. 11362-11366, May, 2016

ISSN: 0976-3031

International Journal of Recent Scientific Research

Research Article

FORBIDDEN SUBGRAPHS FOR VICT GRAPH OF A GRAPH

Muddebihal M.H[1] and Jayashree. B. Shetty*[2]

[1]Department of Mathematics Gulbarga University Kalaburgi-585106 Karnataka India
[2]Department of Mathematics Government First Grade College Humnabad-585330 Karnataka India

ARTICLE INFO

Article History:

Received 16th February, 2016
Received in revised form 24th March, 2016
Accepted 23rd April, 2016
Published online 28th May, 2016

ABSTRACT

In this paper we establish characterizations of graphs whose vict graphs are planar, outerplanar, minimally nonouterplanar, and crossing number one in terms of forbidden subgraphs.

INTRODUCTION

Many graphs which are encountered in the study of graph theory are characterized by a type of configuration or subgraphs they posses. However, there are occasions when such graphs are more easily defined or described by the kind of subgraphs they are not permitted to contain. Such subgraphs are called forbidden subgraphs. Normally a characterization of graphs having a given property by means of "forbidding" a certain family of subgraphs has a great interest due to its practical application. Greenwell and Hemminger [1] characterized graphs with planar line graphs interms of forbidden subgraphs.

Definitions, terms and notations used here are the same as in [4]. The graphs considered are finite, undirected without loops or multiple edges. A graph G is homeomorphic to H, if it is possible to insert vertices of degree two into the edges of H to produce G. The vict graph V_n (G) of a graph G as the graph whose vertex set is the union of the set of vertices and set of cutvertices of G in which two vertices are adjacent if and only if corresponding vertices of G are adjacent or corresponding members of G are adjacent or incident. This concept was first studied by M.H.Muddebihal and Jayashree.B.Shetty [5].

For any plane graph G the inner vertex number i(G) of G is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane. We call the inner vertex number i(G) as Kulli number.
A graph G is said to be minimally nonouterplanar if Kulli number is one or i(G)=1.

Many other graph valued functions in graph theory were studied, for example [2, 3, 6, 7].

The following will be useful in the proof of our results.

Theorem A. The vict graph $V_n(G)$ of a planar graph G is planar if and only if G satisfies the following conditions.

1. G is a tree.

Or

2. G does not contain three mutually adjacent cutvertices with Kulli number zero.

Or

3. G has a block B with Kulli number and no cutvertex of B is adjacent to the Kulli number.

Or

4. Every block of G is either a cycle or an edge in which at least one vertex of an odd cycle is not a cutvertex.

Or

5. G has a cycle C_n , n≥4 together with a diagonal edge joining a pair of vertices of any length which are not cutvertices.

Theorem B. Let G be (p,q) graph. Then vict graph $V_n(G)$ is outerplanar if and only if G is nonseparable outerplanar and G is either a path or a cycle.

Theorem C. The vict graph $V_n(G)$ of a graph G is minimally nonouterplanar if and only if G satisfies the following conditions.

1. G is a block with Kulli number one.
   Or
2. G is a path P_n, (n≥3) together with an endedge adjoined to any nonend vertex of a path P_n.
   Or
3. G has a triangle together with a path P_n, (n≥2) adjoined to any vertex of a triangle.

*Corresponding author: Jayashree.B.Shetty
Department of Mathematics Gulbarga University Kalaburgi-585106 Karnataka India

Theorem D. The vict graph V_n (G) of a graph G has crossing number one if and only if G is planar and (1) or (2) or (3) or (4) holds.

1. G contains three mutually adjacent cutvertices with Kulli number zero.
   Or
2. G contains an odd cycle C_n ,(n≥3) and each vertex of C_n is a cutvertex.
   Or
3. G has a block B with Kulli number and a cutvertex of B is adjacent to the Kulli number.
   Or
4. G has a cycle C_n, n≥4 together with a diagonal edge joining a pair of vertices of any length which are cutvertices.

Theorem E. The vict graph V_n (G) of $K_{3,3}$ and K_5, together with an endedge adjoined to any vertex of $K_{3,3}$ and K_5 has crossing number at least two and three.

RUSULTS

We now establish a characterization of graphs whose vict graphs are planar in terms of forbidden subgraphs by using Theorem A.

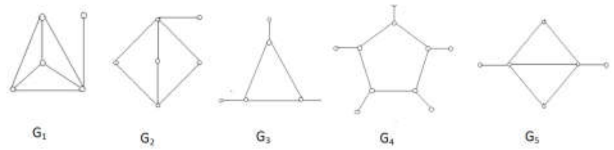

Figure-1

Proof. Let G be a graph with a planar vict graph. We now show that all graphs homeomorphic to K_5 or $K_{3,3}$ and any one of the graphs of Fig1, have nonplanar vict graphs. It follows from Theorem A, since graphs homeomorphic to K_5 or $K_{3,3}$ are nonplanar, graphs homeomorphic to G_1 or G_2 have a block B with Kulli number and a cutvertex of B is adjacent to the Kulli number, graphs homeomorphic to G_3 have three mutually adjacent cutvertices with Kulli number zero, graphs homeomorphic to G_4 have a cycle homeomorphic to C_{2n+1} and each vertex of C_{2n+1} is a cutvertex, graphs homeomorphic to G_5 have a cycle C_4 together with a diagonal edge joining a pair of vertices of any length which are cutvertices.

Conversely, suppose that G contains no subgraphs homeomorphic to either K_5 or $K_{3,3}$ or any one of the graphs of Fig1. Assume that G is a block B which is nonplanar. Then G contains a subgraph homeomorphic to either K_5 or $K_{3,3}$, a contradiction. Hence G is planar.

Suppose G does not contain Kulli number and G is not a tree. Then we consider the following cases.

Case1. Assume G has a subgraph H such that H has three vertices v_1 ,v_2 and v_3 such that v_2 , $v_3 \in N(v_1)$; v_1 , $v_3 \in N(v_2)$ and v_1 , $v_2 \in N(v_3)$ .Then G has a subgraph homeomorphic to G_3 , a contradiction.

Case2. Assume G has a cycle C_{2n+1}. Then we consider the subcases of the case 2.

Subcase 2.1. Suppose for a cycle C_{2n+1}, each vertex of C_{2n+1} is a cutvertex. Then for n=1, G has a subgraph homeomorphic to G_3, a contradiction.

Subcase 2.2. Suppose for a cycle C_{2n+1} with n=2. Then G has a subgraph homeomorphic to G_4, a contradiction.

Case 3. Assume G has a cycle C_n, (n ≥4) there exist two vertices v_1 and v_2 of C_n which are cutvertices and are joined by a path of length one. Then G has a subgraph homeomorphic to G_5, a contradiction.

In each case, we arrived at a contradiction. Hence G has no Kulli number, suppose G has a Kulli number. Then we have the following cases.

Case 1. Assume G has cyclic block such that all vertices of this block lie on exterior region of this block. Then there exists a vertex w lie in the interior of a cyclic block and three distinct vertices v_1, v_2, v_3 lie on cycle. If there exists three mutually edge disjoint paths joining w and distinct vertices v_1, v_2, v_3 of a cycle. Further either v_1 or v_2 or v_3 is a cutvertex. Then G has a subgraph homeomorphic to G_1 , a contradiction.

Case 2. Assume G has a cyclic block and embedded in a plane so that exterior region is bounded by a cycle C. Since not all vertices of G lies on C, there are one or more vertices lying interior to C. If there exists a vertex w interior to C and two disjoint paths between w and two distinct vertices u and v of C. From the choice of C, the edge uv does not belongs to C. Further either u or v is a cutvertex. Then G has a subgrpah homeomorphic to G_2, a contradiction.

We have exhausted all possibilities. In each case, we found that G contains one of our forbidden subgraphs. Thus by Therorem A, G has a planar vict graph.

The following Theorem we characterize those graphs whose vict graphs are outerplanar in terms of forbidden subgraphs.

Theorem 2. A graph G has an outerplanar vict graph if and only if it has no subgraphs homeomorphic to K_4 or $K_{2,3}$ or W_n , n ≥5 or $K_{1,3}$.

Proof. Let G be a graph with an outerplanar vict graph. We now show that all graphs homeomorphic to either K_4 or $K_{2,3}$ or Wn, n≥5 or $K_{1,3}$ have a nonouterplanar vict graph.

It follows from Therem B, since graphs homeomorphic to K_4 or $K_{2,3}$ or Wn, n≥5 is a block with Kulli number, graphs homeomorphic to $K_{1,3}$ have a vertex of degree 3.

Conversely, suppose G contains no subgraph homeomorphic to K_4 or $K_{2,3}$ or $K_{1,3}$ or Wn , n ≥5. Now assume that (G) ≥4. Then G contains a subgraph homeomorphic to Wn, n ≥5, a contractdiction.Hence (G)≤3.

Let v be a cutvertex of G with deg v =3. We prove that v is a cutvertex. If not, let a, b and c be the vertices of G adjacent to v. Then there exists paths between every pair of vertices a, b, c not containing v. G has a subgraph homeomorphic to K_4, a contradiction. Further if v lies on 3 blocks of G, then G has a subgraph homeomorphic to $K_{1,3}$, a contradiction. Suppose there exist two vertices v_1 and v_2 of G with degree 3 which are not cutvertices . Then there exist an edge disjoint paths of length at least two joining v_1 and v_2. Hence G has a subgraph homomorphic to $K_{2,3}$, a contradiction. Thus by Theorem B, it implies that G has an outerplanar Vict graph.

We now characterize those graphs whose Vict graphs are minimally nonouterplanar in terms of forbidden subgraphs.

Theorem3: The vict graph $V_n(G)$ of a graph G is minimally nonouterplanar if and only if it has no subgraph homeomorphic to $K_{1,4}$, $K_{2,4}$, $K_{3,3}$-C_4 and also it fails to contain a subgraph homeomorphic to one of the graphs of Fig. 2.

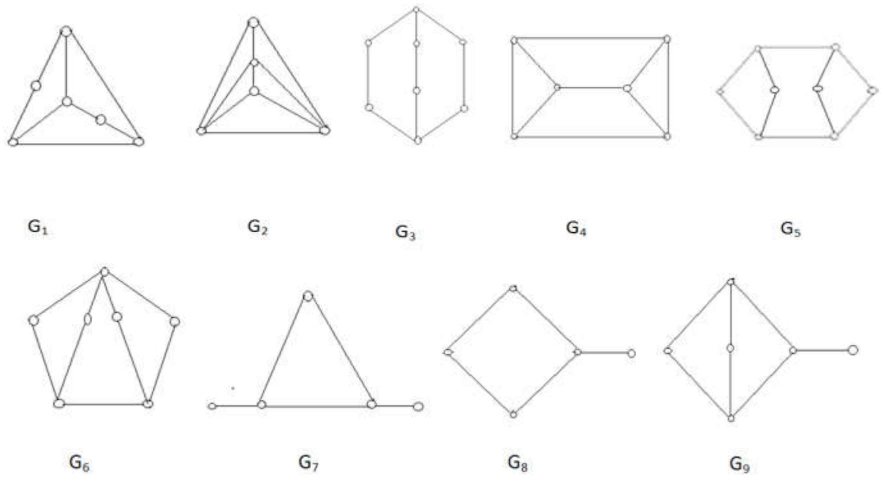

Figure 2

Proof: Let G be a graph for which the vict graph $V_n(G)$ is nonouterplanar. Now suppose that $V_n(G)$ is minimally nonouterplanar. We now show that all graphs homeomorphic to $K_{1,4}$, $K_{2,4}$, $K_{3,3}$- C_4 and also one of the graphs of Fig. 2, with respect to the nonseperable or cutvertices have no minimally nonouterplanar vict graphs. For any graph G, $V_n(G)$ G if and only if G is a block. It follows from Theorem C, since graphs homeomorphic to $K_{1,4}$ have a cutvertex of degree 4, graphs homeomorphic to $K_{3,3}$- C_4 have two cutvertices of degree 3, graphs homeomorphic to $K_{2,4}$, G_1 or G_2 or G_3 or G_4 or G_5 or G_6 they all have nonseperable with Kulli number 2, graphs homeomorphic to G_7 have a triangle together with two edges adjoined to different vertices of a triangle, graphs homeomorphic to G_8 have a cycle C_4 together with an endedge adjoined to any vertex of a cycle C_4 , graphs homeomorphic to G_9 have $K_{2,3}$ together with an endedge adjoined to a vertex which is not adjacent to Kulli number.

Conversely, suppose that G contains no subgraph homeomorphic to $K_{1,4}$, $K_{2,4}$, $K_{3,3}$- C_4 and also one of the graphs of Fig.2, with respect to the cutvertices. Assume G has no Kulli number one. Since G is not outerplanar, it has Kulli number k ($k\geq 2$). Then it must contain more than four vertices. Embed G in the plane so that a maximum number of vertices lie on the exterior cycle C. Since G has no cutvertex and Kulli number greater or equal to two, there are at least two vertices lie in the interior of G. Let v_1 and v_2 be the vertices interior to C which are respectively adjacent to vertices u_1 and u_2 on C. G is a block and deg $v_i \geq 2$, i = 1, 2. Then (1) there is a path P from v_1 to v_2, (2) There are two non-crossing paths P_i , i=1,2 or (3) there are two crossing paths P_i , i = 1, 2.

Now we consider the following cases.

Case 1. Assume there is a path P from v_1 to v_2. Then there are two subcases to consider.

Subcase 1.1. Suppose the vertices u_1 and u_2 are consecutive on C. In this case, some vertex of P must have degree at least 3. Otherwise, the path P could be transferred outside of C to produce a planar embedding of G having a longer exterior cycle. Then, there is a path from a vertex x of P, to a vertex u_3 of C not containing any other vertex of P and u_3 is adjacent with one of u_i say u_2. edges of C and the three paths from x to C induce a subgraph homeomorphic from G_1 (see Fig. 3).

The Further more if u_3 is also adjacent with u_1, then there exist two more paths-from v_1 (or v_2) to u_3 and from x to u_1 (or u_2). Otherwise the path P could be transferred outside of C to produce a minimally nonouterplanar embedding of G. The edges of C, the three paths from x to C, the two paths from v_1 to C and the path from v_1 to x induce a subgraph homeomorphic from G_2 (see Fig. 4).

Subcase 1. 2. Suppose the vertices u_1 and u_2 are not consecutive on C. Then the length of each path from u_1 to u_2 on C is at least three. Otherwise the path P can be transferred outside of C to produce a minimally nonouterplanar embedding of G. Clearly, the edges of C and those of the path through v_1, v_2 from u_1 to u_2 induce a subgraph homeomorphic from G_3 (see Fig5).

Case 2. Assume there are two paths P_i, i=1,2 from v_i to some vertex w_i on C. Then we consider two subcases of case 2.

Subcase 2.1. Suppose the vertices u_i and w_i, i=1, 2 are consecutive on C. Then in this case, some vertex of P_i, i=1, 2 different from w_i, must have degree at least 3. Otherwise, the paths P_i, i=1,2 can be transferred outside of C to produce an outerplanar embedding of G. Hence, there is a path from a vertex x of P_1 to a vertex y of P_2. Then the edges of C and the paths from the vertices x, y to C induce a subgraph homeomorphic from G_4 (see Fig. 6).

Subcase 2.2. Suppose the vertices u_i and w_i, i=1, 2 are not consecutive on C. Then clearly, the edges of C and those of the paths through v_i, i = 1, 2 from u_i to w_i , induce a subgraph homeomorphic from G_5 (see Fig. 7). Further if $u_1 = u_2$, then the edges of C and those of the paths through v_i, i=1,2 from u_i to w_i induce a subgraph homeomorphic from G_6 (see Fig. 8). Furthermore if we put $w_1 = w_2$, then the edges of C and those of the paths through v_i , i =1,2 from u_1 to w_1 induce a subgraph homeomorphic from $K_{2,4}$ (see Fig 9).

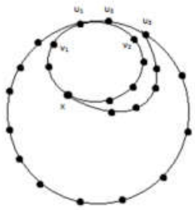

Fig.3

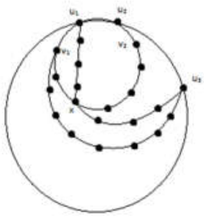

Fig.4

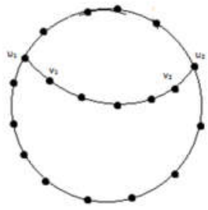

Fig.5

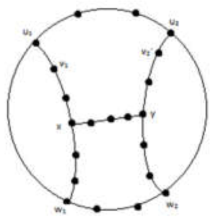

Fig.6

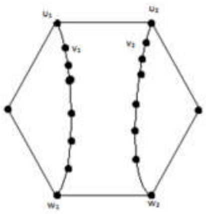

Fig.7

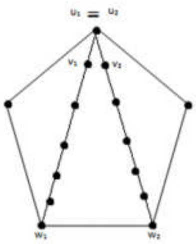

Fig.8

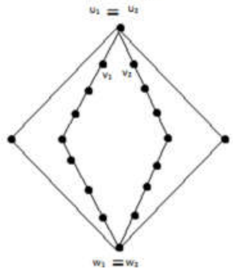

Fig.9

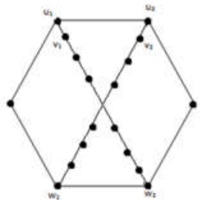

Fig.10

Case 3. Assume there are two paths P_1 (v_1— w_2) and P_2 (v_2 — w_1) where w_1, w_2 are on C. Then the edges of C and those of the paths u_1— w_2 and u_2 —w_1 through v_1 and v_2 respectively induce

a subgraph homeomorphic from G_1 (see Fig. 10).
From the above cases we conclude that G is a block with Kulli number one.

Now assume (G) =4. Then we consider the following cases.

Case4. Assume G has a cutvertex of degree 4. Then it has a subgraph homeomorphic to $K_{1,4}$, a contradiction.

Subcases 6.1. Suppose G has path P_n together with two endedges adjoined to some non end vertices of a path P_n.Then G has a subgraph homeomorphic to $K_{3,3}$- C_4 , a contradiction. Thus G contains a path P_n ($n \geq 3$) together with an endedge adjoined to some nonend vertex of a path P_n ($n \geq 3$).

Subcase 6.2.Suppose G has a triangle together with two edges adjoined to different vertices of a triangle. Then G has a subgraph homeomorphic to G_7, a contradiction.

Case 7: Assume G has a unique cutvertex of degree 3. Then we consider the following subcases of case 7.

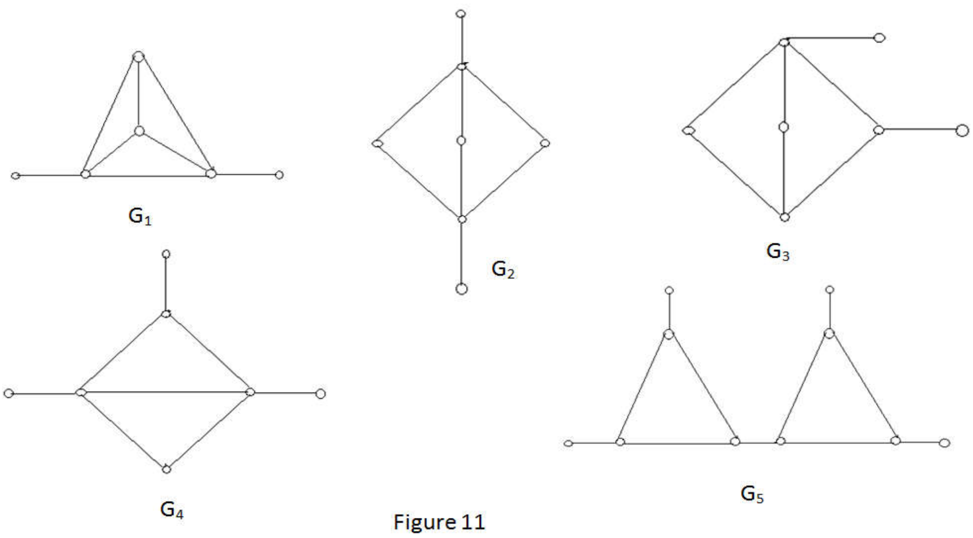

Figure 11

Case5. Assume G has two vertices of degree 4 which are not cutvertics. Then G has a subgraph homeomorphic to $K_{2,4}$, a contradiction. Thus G contains the cutvertices of degree 3.

In the following cases we discuss the number of cutvertices of degree 3.

Case6. Assume G has two cutvertices of degree3. Then we consider the following subcases of case 6.

Subcase 7.1*:* Suppose a cutvertex lies on two blocks in which one block is isomorphic to a cycle C_4 and other is an edge. Then G has a subgraph homeomorphic to G_8 , a contradiction.

Subcase 7.2: Suppose G has a subgraph as $K_{2,3}$ and an endedge adjoined to it. Then G has a subgraph homeomorphic to G_9, a contradiction.

From the above cases we conclude that G is a block with Kulli number one, and no vertex of this block is a cutvertex.

Thus by Theorem C, it implies that G has a minimally nonouterplanar vict graph.

We now characterize of graphs whose vict graphs have crossing number one, in terms of forbidden subgraphs.

***Theorem 4**:* A graph has a vict graph with crossing number one if and only if it has no subgraph homeomorphic to (i) $K_{3,3}$ or K_5 together with an endedge adjoined to any vertex of $K_{3,3}$ or K_5 or (ii) G has a cycle C_n ($n \geq 6$) together with an diagnol edge joining a pair of vertices of length (n-4) and each vertex of a cycle C_n is a cutvertex and also it fails to contain a subgraphs homeomorphic to one of the graphs of Fig.11 with respect to the cutvertices.

***Proof**:* Assume G is a connected plane graph whose vict graph Vn(G) has crossing number one. We prove that all graphs homeomorphic to condition (i) or (ii) or G_1 or G_2 or G_4 or G_5 have C[Vn(G)]>1, by Theorem D, since graphs homeomorphic to $K_{3,3}$ or K_5 together with an endedge adjoined to any vertex of $K_{3,3}$ or K_5 are nonplanar graphs, graphs homeomorphic to cycle C_n ($n \geq 6$) where n is odd or even together with an diagonal edge joining a pair of vertices of length (n-4) and each vertex of a cycle C_n is a cutvertex , graphs homeomorphic to G_1 or G_3 have block B with Kulli number and two adjacent cutvertices of B are adjacent to the Kulli number, graphs homeomorphic to G_2 has a block B with Kulli number and two nonadjacent cutvertices of B are adjacent to the Kulli number, graphs homeomorphic to G_4 have three mutually adjacent cutvertices with a diagonal edge joining a pair of vertices of length (n-4), graphs homeomorphic to G_5 has two sets of three mutually adjacent cutvertices.

Conversely, assume that G is a connected plane graph and does not contain a subgraph homeomorphic to any one of graphs of condition (i) or condition (ii) or G_1 or G_2 or G_3 or G_4 or G_5. We shall show that G Satisfies (1) or (2) or (3) or (4) and hence by Theorem D, Vn(G) has crossing number one. It implies that G has no subgraph homeomorphic to $K_{3,3}$ and K_5 together with an endedge adjoined to any vertex of $K_{3,3}$ or K_5. Then by Theorem E, G has crossing number at least two and three.

We now consider the following cases.

Case 1. Assume $\triangle$(G) =4. Then we consider the following subcases.

Subcases 1.1. Suppose G contains two cutvertices of degree 4, which are adjacent to Kulli number. Then it has a subgraph homeomorphic to G_1 or G_2 , a contradiction. Further suppose G contains two cutvertices in which one cutvertex is adjacent to Kulli number and also adjacent to a cutvertex. Then G has a subgraph homeomorphic to G_3, a contradiction.

Thus G has a block B with Kulli number and a cutvetex of B is adjacent to the Kulli number.

Subcase 1.2. Suppose every v_i (i=1, 2, 3) lies on 2 blocks in which one block is isomorphic to a cycle and other block is an edge. Then G has a subgraph homeomorphic to G_4 or G_5, a contradiction. Thus G contains three mutually adjacent cutvertices with Kulli number zero.

Subcase 1.3. Suppose every cutvertex v_i and v_{i+4}, $\forall$ i=1 of degree 4 and remaining cutvertices v_i ($\forall$ i=2,3,4,6_ _ _ ,n) where ($n \geq 6$) lie on 2 blocks, such that one block is isomorphic to a cycle C_n and other is an edge. Then G has a subgraph homeomorphic to condition (ii), a contradiction. Thus G contains an odd cycle C_n ($n \geq 3$) and each vertex of C_n is a cutvertex or G has a cycle C_n ($n \geq 4$) together with a diagonal edge joining a pair of vertices of any length which are cut vertices.

Case 2. Now Assume $\triangle$ (G) =3. Then we consider the following subcase.

Subcase 2.1. Suppose every cutvertex v_i (i=1, 2, 3, 4, 5 and 6) lie on two blocks such that one block is isomorphic to cycle and other block is an edge. Then G has a subgraph homeomorphic to G_5, a contradiction. Thus G contains only three mutually adjacent cutvertices with Kulli number zero.
Thus by Theorem D, it implies that $V_n(G)$ has crossing number one.

References

1. D.L. Greenwell and R.L. Hemminger, Forbidden subgraphs for graphs with planar line graph. Discrete Math.2, 31-34, (1972).
2. H.Whitney, Congruent graphs and the connectivity of graphs, Amer. J. Math. 54, 150-168,(1932).
3. J. Sedlacek, Some properties of interchange graphs, Theory of graphs and its applications, Symposium Smolenice, 145-150, (1963).
4. M.Behzad and G. Chartrand, An introduction to the Theory of Graphs, Allyn and Bacon, Inc., Boston, (1972).
5. M.H. Muddebihal and Jayashree.B.Shetty, characterization of vict graph of graph, IRJPA-6(2), 238-246, Feb (2016).
6. O. Ore, The four colour problem, Acad, Press, N.Y. (1967).
7. V.R.Kulli and E. sampathkumar, On the interchange graph of a finite planar graph, J.Indian Math.soc, 37, 339-341 (1973).

****

How to cite this article:

Muddebihal M.H and Jayashree.B.Shetty.2016, Forbidden Subgraphs for Vict Graph of a Graph. *Int J Recent Sci Res.* 7(5), pp. 11362-11366.

ISSN 0976-3031