Available Online at http://www.recentscientific.com

International Journal of Recent Scientific Research
Vol. 8, Issue, 3, pp. 15873-15875, March, 2017

ISSN: 0976-3031

International Journal of Recent Scientific Research

DOI: 10.24327/IJRSR

Research Article

BOUNDS FOR THE ZEROS OF A POLYNOMIAL

M.H. Gulzar

Department of Mathematics, University of Kashmir, Srinagar

DOI: http://dx.doi.org/10.24327/ijrsr.2017.0803.0024

ARTICLE INFO

Article History:

Received 15th December, 2016
Received in revised form 25th January, 2017
Accepted 23rd February, 2017
Published online 28th March, 2017

Key Words:

Coefficients, Polynomial, Zeros.

ABSTRACT

In this paper we find a bound for all the zeros of a polynomial in terms of its coefficients similar to the bound given by Cauchy's classical theorem.

INTRODUCTION

Regarding a bound for all the zeros of a polynomial, Cauchy[1] (see also [5],[6],[8]) proved the following famous result known as Cauchy's Theorem:

Theorem A. All the zeros of the polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n lie in the circle $|z|<1+M$,

where $M=\max_{0\le j\le n-1}\left|\frac{a_j}{a_n}\right|$.

Various generalizations, extensions and improvements of the above result are available in the literature.

An important class of polynomials is that of the lacunary type i.e. of the type

$$P(z)=a_0+a_1z+......+a_pz^p+a_{n_1}z^{n_1}+a_{n_2}z^{n_2}+.......+a_{n_k}z^{n_k},$$

where

$$0<p=n_0<n_1<n_2<......<n_k; a_0a_pa_{n_1}a_{n_2}......a_{n_k}\ne 0$$

, the coefficients $a_j, 0\le j\le p$, are fixed, $a_{n_j}, j=1,2,......,k$ are arbitrary and the remaining coefficients are zero. Landau[3,4] initiated the study of such polynomials in 1906-7 in connection with his study of the Picard's theorem and proved that every trinomial

$$a_0+a_1z+a_nz^n, a_1a_n\ne 0, n\ge 2$$

has at least one zero in $|z|\le 2\left|\frac{a_0}{a_1}\right|$ and every quadrinomial

$$a_0+a_1z+a_mz^m+a_nz^n, a_1a_ma_n\ne 0, 2\le m<n$$

has at least one zero in $|z|\le \frac{17}{3}\left|\frac{a_0}{a_1}\right|$.

Q.G.Mohammad [7] in 1967 proved the following theorem:

Theorem B. All the zeros of the polynomial

$P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n lie in the circle

$$|z|\le \max(L_p, L_p^{\frac{1}{n}})$$

where

$$L_p=n^{\frac{1}{q}}\{\sum_{j=0}^{n}\left|\frac{a_j}{a_n}\right|^p\}^{\frac{1}{p}},$$

*Corresponding author: M.H. Gulzar
Department of Mathematics, University of Kashmir, Srinagar

p>1,q>1with $\frac{1}{p}+\frac{1}{q}=1$.

Gulzar [2] recently proved the following result:

Theorem D. All the zeros of the polynomial $P(z)=\sum_{j=0}^{n}a_jz^j, a_na_{n-1}\neq 0$ of degree n lie in the circle

$$|z|\le \max(L, L^{\frac{1}{n+1}})$$

where

$$L=(n+1)^{\frac{1}{q}}\{\sum_{j=1}^{n}\left|\frac{a_{n-1}a_{n-j}-a_na_{n-j-1}}{a_n^2}\right|^p\}^{\frac{1}{p}},$$

p>1,q>1 and $\frac{1}{p}+\frac{1}{q}=1$.

Main Results

In this paper we prove the following generalization of Theorem D:

Theorem 1. All the zeros of the polynomial $P(z)=\sum_{j=0}^{\lambda}a_jz^j+a_nz^n, a_\lambda a_n\neq 0, 0\le\lambda\le n-1$, of degree n lie in the circle

$$|z|\le \max(L, L^{\frac{1}{n+1}})$$

where

$$L=(n+1)^{\frac{1}{q}}\{\sum_{j=0}^{\lambda}\left|\frac{a_\lambda a_{\lambda-j}-a_na_{\lambda-j-1}}{a_n^2}\right|^p\}^{\frac{1}{p}},$$

p>1,q>1 and $\frac{1}{p}+\frac{1}{q}=1$.

Remark. Choosing $\lambda=n-1$ in Theorem 1, we get the following result which is equivalent to Theorem B:

Corollary 1. All the zeros of the polynomial $P(z)=\sum_{j=0}^{n}a_jz^j, a_na_{n-1}\neq 0$ of degree n lie in the circle

$$|z|\le \max(L, L^{\frac{1}{n+1}})$$

where

$$L=(n+1)^{\frac{1}{q}}\{\sum_{j=0}^{n-1}\left|\frac{a_{n-1}a_{n-j-1}-a_na_{n-j-2}}{a_n^2}\right|^p\}^{\frac{1}{p}}(a_{-1}=0),$$

p>1,q>1 and $\frac{1}{p}+\frac{1}{q}=1$.

Proof of Theorem 1

Consider the polynomial

$$F(z)=(a_\lambda-a_nz)P(z)$$

$$=(a_\lambda-a_nz)(a_nz^n+a_\lambda z^\lambda+......+a_1z+a_0)$$
$$=-a_n^2z^{n+1}-a_\lambda a_nz^{\lambda+1}+(a_\lambda a_\lambda-a_na_{\lambda-1})z^\lambda+(a_\lambda a_{\lambda-1}-a_na_{\lambda-2})z^{\lambda-1}+......$$

$$+......+(a_\lambda a_1-a_na_0)z+a_\lambda a_0$$

$$=-a_n^2z^{n+1}+\sum_{j=0}^{\lambda+1}(a_\lambda a_{\lambda+1-j}-a_na_{\lambda-j})z^{\lambda+1-j}.$$

Hence

$$|F(z)|\ge|a_n^2||z|^{n+1}-\sum_{j=0}^{\lambda+1}|a_\lambda a_{\lambda+1-j}-a_na_{\lambda-j}||z|^{\lambda+1-j}$$

$$=|a_n^2||z|^{n+1}[1-\sum_{j=0}^{\lambda+1}\left|\frac{a_\lambda a_{\lambda+1-j}-a_na_{\lambda-j}}{a_n^2}\right|.\frac{1}{|z|^{n-\lambda+j}}]$$

Applying Holder's inequality, we get

$$|F(z)|\ge|a_n^2||z|^{n+1}[1-(\sum_{j=0}^{\lambda+1}\left|\frac{a_\lambda a_{\lambda+1-j}-a_na_{\lambda-j}}{a_n^2}\right|^p)^{\frac{1}{p}}.(\sum_{j=0}^{\lambda+1}\frac{1}{|z|^{(n-\lambda+j)q}})^{\frac{1}{q}}]$$

Now if $L\ge1$, then $\max(L,L^{\frac{1}{n+1}})=L$. Hence for $|z|\ge1$ so that $|z|^{(n-\lambda+j)q}\ge|z|^{(1+j)q}\ge|z|^q$

i.e. $\frac{1}{|z|^{(n-\lambda+j)q}}\ge\frac{1}{|z|^q}$,

$$|F(z)|\ge|a_n^2||z|^{n+1}[1-(\lambda+2)^{\frac{1}{q}}(\sum_{j=0}^{\lambda+1}\left|\frac{a_\lambda a_{\lambda+1-j}-a_na_{\lambda-j}}{a_n^2}\right|^p)^{\frac{1}{p}}.\frac{1}{|z|}]$$

$$\ge|a_n^2||z|^{n+1}[1-(n+1)^{\frac{1}{q}}(\sum_{j=0}^{\lambda+1}\left|\frac{a_\lambda a_{\lambda+1-j}-a_na_{\lambda-j}}{a_n^2}\right|^p)^{\frac{1}{p}}.\frac{1}{|z|}]$$

$$=|a_n^2||z|^{n+1}[1-\frac{L}{|z|}]>0$$

if

$|z|>L$.

Thus all the zeros of F(z) lie in $|z|\le L$ in this case.

If $L\le1$, then $\max(L,L^{\frac{1}{n+1}})=L^{\frac{1}{n+1}}$. Hence, for $|z|\le1$ so that $|z|^{(n-\lambda+j)q}\ge|z|^{(n+1)q}$

i.e. $\frac{1}{|z|^{(n-\lambda+j)q}}\ge\frac{1}{|z|^{(n+1)q}}$,

$$|F(z)|\ge|a_n^2||z|^{n+1}[1-(\lambda+2)^{\frac{1}{q}}(\sum_{j=0}^{\lambda+1}\left|\frac{a_\lambda a_{\lambda+1-j}-a_na_{\lambda-j}}{a_n^2}\right|^p)^{\frac{1}{p}}.\frac{1}{|z|^{(n+1)q}}]$$

$$\geq |a_n^2||z|^{n+1}[1-(n+1)^{\frac{1}{q}}(\sum_{j=0}^{\lambda+1}\left|\frac{a_\lambda a_{\lambda+1-j}-a_n a_{\lambda-j}}{a_n^2}\right|^p)^{\frac{1}{p}}.\frac{1}{|z|^{n+1}}]$$

$$=|a_n^2||z|^{n+1}[1-\frac{L}{|z|^{n+1}}]$$

$$>0$$

if

$$|z|>L^{\frac{1}{n+1}}.$$

Thus all the zeros of F(z) lie in $|z|\leq L^{\frac{1}{n+1}}$ in this case.

Since the zeros of P(z) are also the zeros of F(z), the theorem follows.

References

1. A.L.Cauchy, Exercises de mathematiques, IV Anne de Bure Freses, 1829.
2. M.H.Gulzar, Bounds for Polynomial Zeros, *International Journal of Advance Foundation and Research in Science and Technology* (IJAFRSE), Vol.1 Issue 12, April 2015,20-25.
3. E.Landau, Ueber den Picardschen satz, Vierteljahrsschrift Naturforsch, Gesellschaft Zirich, 51(1906), 252-318.
4. E.Landau, Sur quelques generalizations du theorem de M.Picard, Ann. EcoleNorm. 24, 3 (1907), 17-201.
5. M.Marden, the Geometry of the Zeros of a Polynomial in a Complex Variable, Math. Surveys No.3, AMS, Providence RI, 1949.
6. G.V.Milovanovic, D.S. Mitriminovic, T.M. Rassias, Topics in Polynomials, Extremal Problems, Inequalities, Zeros, World Scientific Publishing Co.,Singapore,1994.
7. Q.G.Mohammad, Location of the Zeros of Polynomials, Amer. Math. Monthly, 74(1967), 290-292.
8. Q.I.Rahman and G. Schmeisser, Analytic Theory of Polynomials, Oxford University Press Inc., New York, 2002.

****

How to cite this article:

M.H. Gulzar.2017, Bounds For The Zeros of A Polynomial. *Int J Recent Sci Res.* 8(3), pp. 15873-15875.