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Research Article

NEW TYPE OF FIXED POINT THEOREM IN A COMPLETE CONE METRIC SPACE

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ABSTRACT

In this paper we prove a generalized fixed point theorem of contraction mapping on cone metric space.

INTRODUCTION

Huang and Zhang [5] in 2007 generalized the concept of a metric space to cone metric space. They replaced the set of real numbers by an ordered Banach space and obtained some fixed point theorem. In the present paper we shell establish a more generalized fixed point theorem of contractive mapping on cone metric space.

Preliminaries

The following notions have been used to prove the main result.

Definition: Let E be a real Banach Space. A subset P of E is called cone [5] if and only if

1. P is closed, non empty and $P \neq \{o\}$.
2. $a, b \in R$, $a, b \geq 0$, and $x, y \in P$ implies $ax + by \in P$.
3. $x \in P$ and $-x \in P \Rightarrow x = o$.

Definition: The partial ordering [5] $\leq$ with respect to $P \subseteq E$ is defined by $x \leq y$ if and only if $y - x \in P$.

Definition: A cone P is called normal [5] if there is a number k > o such that for all $x, y \in E$, the inequality
$0 \leq x \leq y$ implies $\|x\| \leq k \|y\|$

The least positive number k satisfying the above inequality is called the normal constant [5] of P.

Definition: Let X be a non empty set. Suppose that the mapping $d: X \times X \rightarrow E$ Satisfies

(d_1) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
(d_2) $d(x, y) = d(y, x)$ for $x, y \in X$.
(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in X$.

Then d is called a cone metric [5] on X and (X, d) is called a Cone metric space [5].

Definition: Let (X, d) be a Cone metric space. Then a sequence $\{x_n\}$ is

(a) Cauchy sequence [5] if for every $c \in E$ with $0 << c$, there is N such that for all n, m > N, $d(x_m, x_m) << c$.
(b) Convergent sequence [5] if for every c in E with $0 << c$, there is N such that for all n > N, $d(x_n, x) << c$ for some fixed s in X.

Definition: A cone metric space X is said to be complete [5] if every Cauchy sequences in X is convergent in X.

It is known that (x_n) Converges to $x \in X$. if and only it $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$, Also the limit of a convergent sequence is unique provided P is a normal cone with normal constant k [5].

Fixed Point Theorem

In this section we shell prove the following fixed point theorem of contractive mapping.

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Theorem: Let (X, d) be a complete cone metric space, P be a normal cone with normal constant k. If T_1 and T_2 are two mapping of X into itself such that

$$d(T_1^P(x), T_2^q(y)) \le k_1 d(x, T_1^P(x)) + k_2 d(y, T_2^q(y)).$$

Where $x, y \in X$, p and q are positive integers, $k_1 > 0$, $k_2 > 0$, $k_1 + k_2 < 1$.Then T_1 and T_2 have unique and common fixed point in X.

Proof: Let $x \in X$ be arbitrary. First we define a sequence $\{x_n\}$ in X as follows.

$x_1 = T_1^P(x)$, $x_2 = T_2^q(x_1)$, $x_3 = T_1^P(x_2)$, $x_4 = T_2^q(x_3)$ and so on.

In general, $x_{2n+1} = T_1^P(x_{2n})$ and $x_{2(n+1)} = T_2^q(x_{2n+1})$.

Now $d(x_1, x_2) = d(T_1^P(x), T_2^q(x_1))$

Or $d(x_1, x_2) \le k_1 d(x, T_1^P(x)) + k_2 d(x, T_2^q(x_2))$

$\le k_1 d(x, x_1) + k_2 d(x_1, x_2)$.

Therefore $d(x_1, x_2) \le k_1 d(x, x_1)$

Also $d(x_2, x_3) = d(T_2^q(x_1), T_1^P(x_2))$

Or $d(x_2, x_3) \le k_1 d(x_2, T_1^P(x_2)) + k_2 d(x_1, T_2^q(x_1))$.

$\le k_1 d(x_2, x_3) + k_2 d(x_1, x_2)$.

Thus $d(x_2, x_3) \le \dfrac{k_2}{1-k_1} \dfrac{k_1}{1-k_2} d(x_1, x_2)$

Put $r_1 = \dfrac{k_1}{1-k_2}$, $r_2 =$, We have $d(x_2, x_3) \le r_1 r_2 d(x_1, x_2)$.

s

Since $K_1 + K_2, < 1$, r_1 and $r_2 < 1$ after generalization we have,

$d(x_{2n}, x_{2n+1}) \le r_2 r_1 \ldots\ldots\ldots\ldots r_1 r_2 r_1 d(x, x_1)$

$\le r_2^n r_1^n d(x, x_1)$

Similarly $d(x_{2n+1}, x_{2(n+1)}) \le r_1^{n+1} r_2^n d(x, x_1)$.

Since $d(x_m, x_{m+n}) \le d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \ldots\ldots\ldots\ldots + d(x_{m+n-1}, x_{m+n})$

So for m = 2*l*, we have

$d(x_m, x_{m+n}) \le (r_2^l r_1^l + r_2^l r_1^{l+1} + r_2^{l+1} r_1^{l+1} + \ldots\ldots\ldots\ldots$ n terms$) \times d(x, x_1)$

$\le (r_2^l r_1^l + r_2^l r_1^{l+1} + r_2^{l+1} r_1^{l+1} + \ldots\ldots\ldots\ldots$ up to infinity$) \times d(x, x_1)$

$= \{r_2^l r_1^l (1 + r_1 r_2 + r_1^2 r_2^2 + \ldots\ldots) + r_1^{l+1} r_2^l (1 + r_1 r_2 + r_1^2 r_2^2 + \ldots\ldots)\} \times d(x, x_1)$.

$= r_2^l r_1^{l+1} (1 + r_1) \dfrac{1}{1 - r_1 r_2} \times d(x, x_1)$

$\to \infty$ as $l \to \infty$ i. e. $m \to \infty$.

Similarly, for $m = 2l + 1$

$d(x_m, x_{m+n}) \le (r_2^l r_1^{l+1} + r_2^{l+1} r_1^{l+1} + r_2^l r_1^{l+2} + \ldots\ldots\ldots\ldots) \times d(x, x_1)$

$= \{r_2^l r_1^{l+1} (1 + r_1 r_2 + r_1^2 r_2^2 + \ldots\ldots) + r_1^{l+1} r_2^{l+1} (1 + r_1 r_2 + r_1^2 r_2^2 + \ldots\ldots)\} \times d(x, x_1)$.

$= r_2^l r_1^{l+1} (1 + r_2) \dfrac{1}{1 - r_1 r_2} \times d(x, x_1)$

$\to \infty$ as $l \to \infty$ i. e. $m \to \infty$.

This shows that $\{x_n\}$ is a Cauchy sequence. Since the space is complete there exists $x_0 \in X$ such that $\lim_{n\to\infty} x_n = x_0$.

We first show that $T_1^P(x_0) = T_2^q(x_0) = x_0$.

Since $d(x_0, T_1^P(x_0)) \le d(x_0, x_t) + d(x_t, T_1^P(x_0))$

$\le d(x_0, x_t) + d(T_2^q(x_{t-1}), T_1^P(x_0))$,where t is taken to be even.

Hence $d(x_0, T_1^P(x_0)) \le d(x_0, x_t) + k_1 d(x_0, T_1^P(x_0)) + k_2 d(x_{t-1}, T_2^q(x_{t-1}))$

Or $(1 - k_1) d(x_0, T_1^P(x_0)) \le d(x_0, x_t) + k_2 d(x_{t-1}, x_t)$

The expression on the right hand side can be made arbitrarily small by choosing t sufficiently large. Therefore, $d(x_0, T_1^P(x_0)) = 0$

i.e. $T_1^P(x_0) = x_0$. Similarly $T_1^P(x_0) = x_0$.

Now we show that x_0 is unique. For suppose that y_0 also satisfies

$T_1^P(y_0) = T_2^q(y_0) = y_0$.

Then $d(x_0, y_0) = d(T_1^P(x_0), T_2^q(y_0))$

$\le k_1 d(x_0, T_1^P(x_0)) + k_2 d(y_0, T_2^q(y_0))$.

So $x_0 = y_0$.

Finally we prove that x_0 is the common fixed point of T_1 and T_2.

For, $T_1(x_0) = x_0 \Rightarrow T_1^P(T_1(x_0)) = T_1(x_0)$.

$\Rightarrow T_1^P(x_0) = x_0$. Since T_1^P has a unique fixed point x_0.

Similarly $T_2(x_0) = x_0$.

Also x_0 is the only fixed point of T_1 and T_2. For suppose if possible $z_0 \ne x_0$ and $T_1(z_0) = T_2(z_0) = z_0$

Then $d(x_0, z_0) = d(T_2(x_0), T_1(z_0))$

$= d(T_2^q(x_0), T_1^P(z_0))$

$\le k_1 d(z_0, T_1^P(z_0)) + k_2 d(x_0, T_2^q(x_0)) = 0$.

Which implies $x_0 = z_0$.

This completes the proof of the theorem.

Remark: The theorem proved by Huang L. G. and Zhang X. is a particular case of above theorem if we take q = p = 1; $k_1 = k_2$ and $T_1 = T_2$.

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