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Research Article

COMPARITIVE STUDY OF A IN HOMOGENEOUS NONLINEAR DIFFUSION EQUATION BY SIMILARITY METHOD AND ITS EXACTEDNESS

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ABSTRACT

Nonlinear partial differential equations are widely used to describe complex phenomena in various fields of science, it is important to seek their exact solutions. The classical Lie symmetry method can be used to find similarity solutions systematically. The motivation for the present study is to carry over these techniques, either singly or collectively for obtaining the nonlinear diffusion equation and its symmetry reductions, namely, the second-order nonlinear ordinary differential equations via the isovector approach. The fundamental basis of the techniques is that, when a differential equation is invariant under a Lie group transformations, a reduction transformation exists. The machinery of the Lie group theory provides a systematic method to search for these special group invariant solutions. In this work, I introduce and proved a reduction theorem that will help us to make some critical reduction answer without having to do any teadious calculation

INTRODUCTION

Definition: Let $\bar{x} = \left(x_1, x_2 .... x_n\right)$ lie in region $D \subset R^n$. The set of transformations $x^* = X(\bar{x}, \varepsilon)$, defined for each $\bar{x}$ in D, depending on parameter ε lying in $S \subset R,$ with $\phi(\varepsilon, \delta)$ defining a law of composition of parameters ε and δ in S, forms a group of transformations on D if:

1. For each parameter ε in S the transformations are one-to-one, onto D, in particular x^* lies in D.
2. S with the law of composition ϕ forms a group G.
3. $x^* = \bar{x}$ when $\varepsilon = e$, i.e., $X(\bar{x}; \varepsilon) = \bar{x}$.
4. If $x^* = X(\bar{x}; \varepsilon)$, $x^{**} = X\left(x^*; \delta\right)$, then $x^{**} = X\left(\bar{x}; \phi(\varepsilon, \delta)\right)$.
5. ε is a continuous parameter, i.e., S is an interval in R. Without loss of generality $\varepsilon = 0$ corresponds to the identity element ε.
6. X is infinitely differentiable with respect to $\bar{x}$ in D and an analytic function of ε in S.
7. $\phi(\varepsilon, \delta)$ is an analytic function of ε and δ, $\varepsilon \in S$, $\delta \in S$.

Definition: The infinitesimal generator of the one-parameter Lie group of transformations $x^* = X(\bar{x}, \varepsilon)$ is the operator

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$$X=X(\bar{x})=\xi(\bar{x}).\nabla=\sum_{i=1}^{n}\xi_i(\bar{x})\frac{\partial}{\partial x_i},$$

where ∇ is the gradient operator, $\nabla=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},.............\frac{\partial}{\partial x_n}\right);$

for any differentiable function $F(\bar{x})=F(x_1, x_2,.............x_n),$

$$XF(\bar{x})=\xi(\bar{x}).\nabla F(\bar{x})=\sum_{i=1}^{n}\xi_i(\bar{x})\frac{\partial F(\bar{x})}{\partial x_i}.$$

Definition: The total derivative operator is defined by

$$\frac{D}{Dx}=\frac{\partial}{\partial x}+y_1\frac{\partial}{\partial y}+y_2\frac{\partial}{\partial y_1}+..........................+y_{n+1}\frac{\partial}{\partial y_n}+.........$$

For a given differentiable function $F(x,y,y_1,y_2,..................y_\ell),$ we have

$$\frac{D}{Dx}F(x,y,y_1,y_2,.........y_\ell)=F_x+y_1F_y+y_2F_{y_1}+y_3F_{y_2}+........+y_{\ell+1}F_{y_\ell}.$$

Definition

Consider an r-parameter Lie group of transformations

$x^*=X(\bar{x};\varepsilon)$ with infinitesimal generators $\{X_\alpha\},\alpha=1,2,..........r,$
defined by

$$\xi_{\alpha j}(\bar{x})=\left.\frac{\partial x_j^*}{\partial\varepsilon_\alpha}\right|_{\varepsilon=0}=\left.\frac{\partial X_j(\bar{x};\varepsilon)}{\partial\varepsilon_\alpha}\right|_{\varepsilon=0}\quad \alpha=1,2,........r,\ \ j=1,2,......n.$$

and

$$X_\alpha=\sum_{j=1}^{n}\xi_{\alpha j}(\bar{x})\frac{\partial}{\partial x_j},\ \alpha=1,2,........r.$$

The commutator of X_α and X_β is a first-order operator

$$\left[X_\alpha,X_\beta\right]=X_\alpha X_\beta\text{-}X_\beta X_\alpha=\sum_{i,j=1}^{n}\left[\begin{array}{l}\left(\xi_{\alpha i}(x)\frac{\partial}{\partial x_i}\right)\left(\xi_{\beta j}(x)\frac{\partial}{\partial x_j}\right)\\-\left(\xi_{\beta i}(x)\frac{\partial}{\partial x_i}\right)\left(\xi_{\alpha j}(x)\frac{\partial}{\partial x_j}\right)\end{array}\right]$$

$$= \sum_{i,j=1}^{n} \eta_j(x)\frac{\partial}{\partial x_j},$$

where

$$\eta_j(x) = \sum_{i=1}^{n}\left(\xi_{\alpha i}(x)\frac{\partial \xi_{\beta j}(x)}{\partial x_i} - \xi_{\beta i}(x)\frac{\partial \xi_{\alpha j}(x)}{\partial x_i}\right).$$

It immediately follows that

$$\left[X_\alpha, X_\beta\right] = -\left[X_\beta, X_\alpha\right].$$

Exact Solutions to A Nonlinear Diffusion Equation

Consider the radially symmetric nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{1}{x^{N-1}}\frac{\partial}{\partial x}\left(x^{N-1}\, u\frac{\partial u}{\partial x}\right). \tag{1}$$

First we determine the Lie point symmetry vector fields.

$$\text{Let } U = a(x,t,u)\frac{\partial}{\partial x} + b(x,t,u)\frac{\partial}{\partial t} + c(x,t,u)\frac{\partial}{\partial u}, \tag{2}$$

where a, b and c are unspecified functions of x, t and u. We apply the algorithm that provides the symmetry algebra by constructing the prolongation of the vector field U,

$$Pr^2U = U + c^x \partial_{u_x} + c^t \partial_{u_t} + c^{xx}\partial_{u_{xx}}, \tag{3}$$

Where

$$c^x = c_x + c_u u_x - a_x u_x - b_x u_t - a_u u_x^2 - b_u u_x u_t, \tag{4}$$

$$c^t = c_t - a_t u_x + c_u u_t - b_t u_t - a_u u_x u_t - b_u u_t^2, \tag{5}$$

and

$$\begin{aligned} c^{xx} = {} & c_{xx} + 2c_{xu}u_x - a_{xx}u_x - b_{xx}u_t + c_{uu}u_x^2 - 2a_{xu}u_x^2 \\ & - 2b_{xu}u_x u_t - a_{uu}u_x^3 - b_{uu}u_x^2\, u_t + c_u u_{xx} - 2a_x u_{xx} \\ & - 2b_x u_{xt} - 3a_u u_x u_{xx} - b_u u_t u_{xx} - 2b_u u_x\, u_{xt}. \end{aligned} \tag{6}$$

The condition of invariance of the equation (1) is

$$Pr^2U(\Delta)\Big|_{\Delta=0} = 0, \tag{7}$$

$$\text{where } \Delta = u_t - (N-1)x^{-1}u\, u_x - u_x^2 - u\, u_{xx}. \tag{8}$$

From (3) and (7) we get,

$$\begin{aligned} & a\left((N-1)x^{-2}uu_x\right) + c\left(-(N-1)x^{-1}u_x - u_{xx}\right) \\ & + c^x\left(-(N-1)x^{-1}u - 2u_x\right) + c^t - uc^{xx} = 0. \end{aligned} \tag{9}$$

Using (4) – (6) and (8) in (9) we get the following set of determining equations

$$a = a(x), \quad b = b(t), \qquad c = c(u),$$

$$\frac{a(N\text{-}1)}{x^2}-\frac{a_x(N\text{-}1)}{x}+a_{xx}=0, \tag{10}$$

$$\text{-}b_t\text{-}\frac{c}{u}+2a_x=0, \tag{11}$$

$$\text{-}c_u-c_{uu}u+\frac{c}{u}=0. \tag{12}$$

Solve the equations (10) –(12) we get the three symmetry vector fields,

$$B_1=\frac{\partial}{\partial t}\ ,\quad B_2=\text{-}\frac{x}{2}\frac{\partial}{\partial x}-u\frac{\partial}{\partial u}\ ,\quad B_3=x\frac{\partial}{\partial x}+(N+2)t\frac{\partial}{\partial t}-uN\frac{\partial}{\partial u}.$$

These fields form a Lie algebra.

Then the similarity solutions of the form

$$u=t^{\text{-}N/N+2}\ f(\eta), \tag{16}$$

$$\text{where }\eta=xt^{\text{-}1/N+2}. \tag{17}$$

Using (16) and (17) in (1) we get,

$$\frac{\text{-}1}{N+2}\ \eta^N\ f=\eta^{N\text{-}1}f\frac{df}{d\eta}+\alpha, \tag{18}$$

where α is an arbitrary constant of integration.

When $\alpha=0,$ the solution to (18) is easily determined in closed form. Here we shall obtain the general solution to (18) for $\alpha\neq 0$ when N=1 and when N=2.

Exact Solutions: *For N=1:*

Equation (21) is then the Riccati equation

$$\alpha\frac{d\eta}{dg}=\frac{\eta^2}{6}\text{-}g. \tag{22}$$

We write

$\eta=-6\alpha\frac{dq}{dg}.q^{\text{-}1},$ and obtain the linear equation $6\alpha^2\frac{d^2q}{dg^2}=gq,$

with general solution

$$q=aAi\left(\left(6\alpha^2\right)^{\text{-}1/3}g\right)+b\ Bi\left(\left(6\alpha^2\right)^{\text{-}1/3}g\right),$$

where a and b are arbitrary constants, and Ai and Bi are Airy functions.

Similarity Solution of the In Homogeneous Nonlinear Diffusion Equation

The similarity reduction of the inhomogeneous nonlinear diffusion equation

$$x^p\frac{\partial u}{\partial t}\ (x,y,t)\ =\ \frac{\partial}{\partial x}\ \left(x^m\ u^n\ u_x\right)+\lambda\frac{\partial}{\partial y}\ \left(y^{\ell}\ u^q\ u_y\right), \tag{1}$$

where p,q,*l*,m and n are arbitrary constants , λ is a parameter. First we determine the Lie point symmetry vector fields. Let

$$U=a(x,y,t,u)\frac{\partial}{\partial x}+b(x,y,t,u)\frac{\partial}{\partial y}+c(x,y,t,u)\frac{\partial}{\partial t}+d(x,y,t,u)\frac{\partial}{\partial u}, \qquad (2)$$

where a,b,c and d are unspecified functions of x,y,t and u. We apply the algorithm that provides the symmetry algebra by constructing the prolongation of the vector field U.

$$Pr^2U=U+d^x\partial_{u_x}+d^y\partial_{u_y}+d^t\partial_{u_t}+d^{xx}\partial_{u_{xx}}+d^{yy}\partial_{u_{yy}}, \qquad (3)$$

where

$$d^x=d_x+(d_u-a_x)u_x-a_uu_x^2\text{-}b_xu_y\text{-}b_uu_xy_y-c_xu_t-c_uu_xu_t \qquad (4)$$

$$d^y=d_y+(d_u-b_y)u_y-a_yu_x-a_uu_xu_y-b_uu_y^2-c_yu_t-c_uu_yu_t, \qquad (5)$$

$$d^t=d_t+(d_u-c_t)u_t-a_tu_x\text{-}a_uu_xu_t-b_tu_y-b_uu_tu_y-c_uu_t^2, \qquad (6)$$

$$\begin{aligned}d^{xx}=&d_{xx}+(2d_{ux}-a_{xx})u_x+u_{xx}(d_u-2a_x)-c_{xx}u_t\\&+u_x^2(d_{uu}-2a_{xu})-3a_uu_xu_{xx}-2c_{xu}u_xu_t\text{-}a_{uu}u_x^3\\&\text{-}c_{uu}u_x^2u_t-2b_xu_{xy}\text{-}2b_uu_xu_{xy}\text{-}b_{xx}u_y-2b_{xu}u_xu_y\\&-b_{uu}u_x^2u_y-b_uu_{xx}u_y\ \text{-}2c_xu_{xt}\text{-}2c_uu_xu_{xt}\text{-}c_uu_{xx}u_t,\end{aligned} \qquad (7)$$

and

$$\begin{aligned}d^{yy}=&d_{yy}+(2d_{uy}-b_{yy})u_y+u_{yy}(d_u-2b_y)-c_{yy}u_t\\&+u_y^2(d_{uu}-2b_{uy})-3b_uu_yu_{yy}-2c_{uy}u_yu_t\text{-}b_{uu}u_y^3\text{-}c_{uu}u_y^2u_t\\&-2a_yu_{xy}\text{-}2a_uu_yu_{xy}\text{-}a_{yy}u_x-2a_{uy}u_xu_y-a_{uu}u_y^2u_x\\&-a_uu_{yy}u_x\ \text{-}2c_yu_{yt}\text{-}2c_uu_yu_{yt}\text{-}c_uu_{yy}u_t.\end{aligned} \qquad (8)$$

The condition of invariance of the equation (1) is

$$Pr^2U(\Delta)\Big|_{\Delta=0}=0, \qquad (9)$$

From (3) and (9) we get,

$$\begin{aligned}&a\begin{pmatrix}\text{-}m(m\text{-}1\text{-}p)x^{m\text{-}2\text{-}p}u^nu_x\text{-}\,n(m\text{-}p)\,u^{n\text{-}1}\,x^{m\text{-}p\text{-}1}u_x^2\text{-}(m\text{-}p)x^{m\text{-}p\text{-}1}u^nu_{xx}\\+p\lambda\ell y^{\ell\text{-}1}u^qu_yx^{\text{-}p\text{-}1}+p\lambda qu^{q\text{-}1}y^{\ell}u_y^2x^{\text{-}p\text{-}1}+p\lambda y^{\ell}u^qu_{yy}x^{\text{-}p\text{-}1}\end{pmatrix}\\&+b\left(\text{-}\lambda\ell(\ell\text{-}1)y^{\ell\text{-}2}u^qu_yx^{\text{-}p}\text{-}\lambda q\ell u^{q\text{-}1}y^{\ell\text{-}1}u_y^2x^{\text{-}p}\text{-}\lambda\ell y^{\ell\text{-}1}u^qu_{yy}x^{\text{-}p}\right)\\&+d\begin{pmatrix}\text{-}mnx^{m\text{-}1\text{-}p}u^{n\text{-}1}u_x\text{-}n(n\text{-}1)u^{n\text{-}2}x^{m\text{-}p}u_x^2\text{-}nx^{m\text{-}p}u^{n\text{-}1}u_{xx}\\\text{-}\lambda q\ell y^{\ell\text{-}1}u^{q\text{-}1}u_yx^{\text{-}p}\text{-}\lambda q(q\text{-}1)u^{q\text{-}2}y^{\ell}u_y^2x^{\text{-}p}-\lambda qy^{\ell}u^{q\text{-}1}u_{yy}x^{\text{-}p}\end{pmatrix}\end{aligned}$$

$$+d^{x}\left(-mx^{m-1-p}u^{n}-2nu^{n-1}x^{m-p}u_{x}\right)+d^{y}\left(-\lambda ly^{\ell-1}u^{q}x^{-p}-2\lambda qu^{q-1}y^{\ell}u_{y}x^{-p}\right)$$
$$+d^{t}+d^{xx}\left(-x^{m-p}u^{n}\right)+d^{yy}\left(-\lambda y^{\ell}u^{q}x^{-p}\right)=0.$$

Using (4)-(8) and (9) in the above equation we get the following set of determining equations

$$a_{xx}-\frac{ma_{x}}{x}+\frac{am}{x^{2}}=0, \tag{10}$$

$$\frac{nd}{u^{2}}-\frac{nd_{u}}{u}-d_{uu}=0, \tag{11}$$

$$-\frac{a(m-p)}{x}-\frac{dn}{u}+2a_{x}-c_{t}=0, \tag{12}$$

$$\frac{2q}{u}b_{y}+\frac{q}{u}d_{u}-d_{uu}+\frac{dq}{u^{2}}=0, \tag{13}$$

$$\frac{2d_{u}}{y}+\frac{b}{y^{2}}+\frac{b_{y}}{y}+\ell b_{yy}=0. \tag{14}$$

Solve the equations (10)-(14), we get

$$a=\frac{2c_{2}}{p-m+2}\,x-\frac{nc_{1}}{p-m+2}x,\qquad b=c_{1}y,\quad c=2c_{2}t+c_{3},\quad d=-c_{1}u,$$

where c_1, c_2 , c_3 are arbitrary constants and $p-m+2=v$. Then the similarity solution and the Similarity variable are given by

$$u=x^{\frac{v}{n}}F(s),\qquad s=t+yx^{\frac{v}{n}}.$$

CONCLUSION

In this paper I have introduced a new theorem studied the solutions of nonlinear diffusion equations by applying Lie symmetry method and derived the exact solutions of a similarity solution of the inhomogeneous nonlinear diffusion equation and compared the results in various cases.

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