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Research Article

EXISTENCE AND UNIQUENESS OF NONLINEAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS BY USING LIPSCHITZ CONDITION

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ABSTRACT

In this paper, we discussed Existence and Uniqueness Theorem for nonlinear second order ordinary differential equations by using Lipschitz condition .Contraction Mapping Principle is used for proving Existence Theorem.

INTRODUCTION

Differential equation occur in connection with numerous problem that are encountered in the various branches of science and engineering. We indicate a few such problems, which could easily be extended.

The method is, procedure for obtaining explicit and implicit formulas for the solution of various ordinary differential equations and the second one is by using basic existence and uniqueness theorem. For illustration some examples are presented.[12]

In this paper we proved existence and uniqueness theorem for nonlinear second order ordinary differential equations and discussed some problems with extension of solutions of these equations.

Some applications of differential equations discussed last section. [2]

Definition

A linear ordinary differential equation of order n, in the dependent variable y and the independent variable x, in an equation that is in, or can be expressed in, the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots \dots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = b(x),$$

where a is not identically zero.

Methods of Solution

When we say that we shall solve a differential equation we mean that we shall find one of more of its solutions. it is concerned with various methods of solving differential equations, The method to be employed depends upon the type of differential equation under consideration, and we shall not enter into the details of specific methods here.

When we have solved a differential equation, does this necessarily mean that we have found a formula for the solution? The answer is no comparatively few differential equations have solutions so expressible; in fact, a closed form solutions is really a luxury in differential equation.

We shall consider certain types of differential equations that do have such closed form solutions and study the exact methods are unavailable. such for finding these desirable solution But we have just noted, such equations are actually in the minority and we much consider what it means to solve equation for which exact methods are unavailable. Such equations are solved approximately by various methods. Among such methods are series methods and graphical methods.

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These methods are not so desirable as exact methods because of the amount of work involved in them and because the results obtained from them one only approximate; but if exact methods are not applicable, one has no choice but to turn to approximate methods.

Modern science and engineering problems continue to give rise to differential equation to which exact methods do not apply and approximate methods are becoming increasing by more important.

Local Existence Theorem: [11]

Let $\mathrm{I} \subseteq \mathrm{R}$ be an open interval$\mathrm{G} \subseteq \mathrm{R}^n, n \geq 1$, be a domain.

Definition: Let us consider a function$f : \mathrm{I} \times \mathrm{G} \to \mathrm{R}^n$. The general form of an explicit ordinary second-order differential equation is as follows:

$$\frac{dv}{du} = f(u, v). \tag{3}$$

Here, we observe that function f gives a Differential Equation on $\mathrm{I} \times \mathrm{G}$., f is also known as the right hand-side of the Differential Equation (3). Here u is the independent variable and v is the dependent variable.

Proposition 1

Consider the function $f : \mathrm{I} \times \mathrm{G} \to \mathrm{R}^n$ which is continuous and consider the equation,

$$\begin{cases} \dfrac{d^2v}{du^2} = f(u, v, v'), \\ v'(u_0) = v'_0, \qquad (u_0, v'_0) \in \mathrm{I} \times G \end{cases} \tag{4}$$

Also, Let $\mathrm{I}_0 \subseteq \mathrm{I}$ be a neighbourhood of u_0. Then, the function $\varphi : \mathrm{I}_0 \subseteq \mathrm{I} \to G$ is a solution of the Cauchy problem (2) if and only if φ is continuous and satisfies the relationship:

$$\varphi'(u) = v'_0 + \int_{u_0}^{u} f(s, \varphi'(s))\, ds, \forall u \in \mathrm{I}_0, \tag{5}$$

called the Integral Equation (4).

Proof: Suppose that $\varphi : \mathrm{I}_0 \subseteq \mathrm{I} \to \mathrm{G}$ is a solution of (4), then, $\varphi$is differentiable, satisfying the equation, that is,

$\frac{d^2\varphi}{du^2} = f(u, \varphi'(u))$, for all $u \in \mathrm{I}_0$,

Hence, φ and the function $u \to \int_{u_0}^{u} (s, \varphi'(s))\, ds$ are the primitives of the same function, $\exists\ \mathrm{C} \in \mathrm{R}^n$s. t.

$$\varphi'(u) = \mathrm{C} + \int_{u_0}^{u} (s, \varphi'(s))\, ds.$$

Because $\varphi'(u_0) = v'_0$, we obtain $\mathrm{C} = v'_0$and that φ'satisfies the equation (5).

In converse part, suppose that φ' is continuous and satisfying the equation (5), then, φ' is differentiable and

$$\begin{cases} \dfrac{d^2\varphi}{du^2} = f(u, \varphi'(u)), \\ v'(u_0) = v'_0. \end{cases}$$

Thus solution of equation (2) is φ'.
Consider the case $n = 1$.

Now we observe that the conditions on the R.H.S. of Differential Equation gives us Local Existence and Uniqueness of solution.

Take a point (u_0, v'_0), in the rectangle

$$\mathrm{D} = \{(u, v) \in \mathrm{R}_2 \mid u_0 - a \leq u \leq u_0 + a, v_0 - b \leq v \leq v_0 + b\} \tag{6}$$

Statement of the Problem

Let $f : \mathrm{D} \to \mathrm{R}$ be a continuous function on D, which satisfies, in D, a Lipschitz condition with respect to its second argument, i.e. there exists $\mathrm{L} > 0$ such that for any $(u, v'_1), (u, v'_2) \in \mathrm{D}$we have

$$|f(u, v'_1) - f(u, v'_2) \leq \mathrm{L}|v'_1 - v'_2| \tag{7}$$

Then, for the equation

$$\frac{d^2v}{du^2} = f(u, v'), \tag{8}$$

There exists a unique solution$v' = v'(u)$, defined for $u_0 - \mathrm{H} \leq u \leq u_0 + \mathrm{H}$, that satisfies the condition $v'(u_0) = v'_0$. Here,

$$\mathrm{H} < min\,(a, \frac{b}{\mathrm{M}}, \frac{1}{\mathrm{L}}),$$

Where

$$\mathrm{M} = max_{\mathrm{D}}\, |f(u, v')|.$$

We know that the limit of successive approximation sequence is the unique solution of the above equation.. The sequence is as follows

$$\begin{cases} v'_0(u) = v'_0, \\ v'_n(u) = v'_0 + \displaystyle\int_{u_0}^{u} f(t, v'_{n-1}(t))\, dt, \quad n \geq 1 \end{cases}$$

Remark: Before giving the proof of following theorem, let us notice that this is a local existence theorem. In fact, one can prove the existence of the desired solution on the interval $u_0 - \mathrm{H} \leq u \leq u_0 + \mathrm{H}$, where $= min\,(a, \frac{b}{\mathrm{M}})$.

Also, notice that instead of asking the Lipschitz condition (3.7) to be fulfilled, one may ask the existence and the boundedness, in the absolute value, in D, of the partial derivative $\frac{\partial^2 f}{\partial v^2}(u, v')$, which is a cruder condition, but a more easily verifiable one.

Lipschitz Condition

The function f_y and its derivatives f_{yy} are continuous in the rectangle T. thus if (x, y_1) and(x, y_2) are points in T, the mean value theorem applies to f_y as a function of y. hence there exists a number y^* between y_1 and $y_1$such that

$$f_y(x, y_1) - f_y(x, y_2) = f_{yy}(x, y^*)(y_1 - y_2)$$

the assumption that f_{yy} is continuous in T allows us to assert that f_{yy} is bounded there. that is, there exists a number $K > 0$ such that

$$|f_{yy}| \leqq K$$

for every point in T.
Since (x, y^*) is in T, it follows that
$|f_y(x, y_1) - f_y(x, y_2)| = |f_{yy}(x, y^*)||y_1 - y_2|$
and$|f_y(x, y_1) - f_y(x, y_2)| \leqq K|y_1 - y_2|$ ………… (24).
for every pair of points (x, y_1) and(x, y_2) in T.

the inequality (24) is called "Lipschitz Condition" for the function f_{yy}

Contraction Mapping Principle

Let (M, d) be a complete metric space and let $\mathrm{A} : \mathrm{M} \to \mathrm{M}$ be a contraction map, i.e. a map for which there exists $a \in (0, 1)$such that

$$d(\mathrm{A}[v'], \mathrm{A}[w]) \leq \alpha d(v', w), \forall v', w \in \mathrm{M}. \quad (9)$$

Then, A has a unique fixed point in M, i.e. there exists$\bar{v} \in \mathrm{M}$ such that $\mathrm{A}[\bar{v}] = \bar{v}$. This point can be found by the method of successive approximations:

$$\bar{v} = \log_{n\to\infty} v'_{n,} \quad (10)$$

Where

$$v'_n = \mathrm{A}[v'_{n\ 1}], n \geq 1 \quad (11)$$

andv'_0 is an arbitrary point in the space M.

Proof of the Theorem

By applying the Contraction-Mapping- Principle, we have to prove Existence and Uniqueness theorem. Consider the space C on which all the functions are continuous on $u_0 - h \leq u \leq u_0 + h$, here $h \leq min\left(a, \frac{b}{\mathrm{M}}\right)$.

On space C we define the distance function as follows,

$$\mathrm{d}(v', \mathrm{w}) = \max_{u_0 - h \leq u \leq u_0 + h} |v'(u) - w(u)|, \quad (12)$$

Thus the space(C, d) is complete, known as the space of uniform convergence, because, the convergence in this metric gives uniform convergence.

Using proposition 3.5.2, we rewrite the equation

$$\begin{cases} \dfrac{d^2v}{du^2} = f(u, v') \\ v'(u_0) = v'_0 \end{cases}$$

as the Integral Equation

$$v'(u) = v'_0 + \int_{u_0}^{u} f(s, v'(s))ds \quad (13)$$

Suppose the operator,

$$\mathrm{A}[v'] = v'_0 + \int_{u_0}^{u} f(s, v'(s))ds, \quad (14)$$

related to function $v'$and $\mathrm{A}[v']$, on $u_0 - h \leq u \leq u_0 + h$ are continuous. But,

$$\left| \int_{u_0}^{u} f(s, v'(s))ds \right| \leq \mathrm{M}h \leq b.$$

Here we prove on the space(C, d), the operator A is a contraction mapping.
Let $v', w \in \mathrm{C}$. Since

$$\mathrm{d}(\mathrm{A}[v'], \mathrm{A}[w]) = \max_{u_0 - h \leq u \leq u_0 + h} \left| \int_{u_0}^{u} (f(s, v') - f(u, w))du \right|,$$

By Lipschitz condition, we obtain,

$$\mathrm{d}(\mathrm{A}[v'], \mathrm{A}[w]) \leq \mathrm{L}\max_{u_0 - h \leq u \leq u_0 + h} \left| \int_{u_0}^{u} (f(u) - w(u))du \right| \leq$$

$$\leq \mathrm{L}h \max_{u_0 - h \leq u \leq u_0 + h} |v'(u) - w(u)| = \mathrm{L}hd(v', w).$$

By taking h s. t. $0 < \mathrm{L}h \leq \alpha < 1$, A satisfying the inequality,

$$d(A[v'], A[w]) \leq \alpha d(v', w), \ \alpha \in (0,1).$$

which implies that A is a contraction mapping on (C, d). Thus, by the Contraction- Mapping- Principle, $\exists \bar{v}$, a unique fixed point of A.

The Integral Equation is,

$$\mathrm{A}[v'] = v',$$

unique fixed point of A is the unique solution of equation (3.8). This solution is obtained by the successive approximation method.

Example

Consider the domain

$$\mathrm{D} = \{(u, v)\ \mathrm{R}_2 |u| \leq ½, |v - 1| \leq 1\}.$$

On D, consider the equation

$$\begin{cases} \frac{d^2v}{du^2} = uv', \\ v'(0) = 1 \end{cases} \quad (15)$$

Calculate third-order approximation of the solution of (15).

here,

$$\frac{\partial^2 f}{\partial v^2}(u, v') = uv''$$

Therefore, on D, we get

$$\frac{\partial^2 f}{\partial v^2}|(u, v') \leq \frac{1}{2}$$

And

$$\mathrm{L} = \max_{(u,v')\in \mathrm{D}} \left| \frac{\partial^2 f}{\partial v^2}(u, v') \right| = \frac{1}{2}.$$

Moreover,

$$\mathrm{M} = \max_{(u,v')\in \mathrm{D}} |f(u, v')| = 1.$$

Hence, $\mathrm{H} < \frac{1}{2}$and the conditions of Theorem 3.5.3 are fulfilled on the interval

$[-\mathrm{H}, \mathrm{H}]$.

The required successive approximations are

$$v'_0(u) = 1, \quad (16)$$

$$v'_1(u) = 1 + \int_0^u u du = 1 + \frac{u^2}{2}, \quad (17)$$

$$v'_2(u) = 1 + \int_0^u \left(1 + \frac{u^2}{2}\right) du = 1 + \frac{u^2}{2} + \frac{u^4}{8} \quad (18)$$

$$v'_3(u) = 1 + \int_0^u u\left(1 + \frac{u^2}{2} + \frac{u^2}{8}\right) du = 1 + \frac{u^2}{2} + \frac{u^2}{8} + \frac{u^6}{48} \quad (3.19)$$

here the error is

$$\varepsilon_3(u) = |v'(u) - v'_3(u)| \leq \mathrm{ML}^3 \frac{|u|^4}{4!}, \text{for } |u| \leq ½.$$

Therefore,

$$\varepsilon_3(u) = \frac{|u|^4}{192}, \text{for} |u| \leq 1/2.$$

Hence,

$$\max_{|u|\leq 1/2} \varepsilon_3(u) \leq \frac{1}{3072}.$$

The solution of equation (14) is

$v'(u) = e^{u^2/2}$.

Here we note that $v'_3(u)$ contains the first four terms in Taylor's series expansion of $e^{u^2/2}$ around the point $u_0 = 0$.

Remark

Let us consider the Cauchy problem

$$\begin{cases} \frac{dv}{du} = f(uv), \\ v(u_0) = v_0 \end{cases} \quad (20)$$

If in a neighbourhood of the point (u_0, v_0) the function f has continuous derivatives up to the order k, then, in some neighbourhood of the point (u_0, v_0), the solution of problem (3.20) has continuous derivatives up to the order $(k + 1)$.

Ex:-1 Show that the sequence of function defined in equation

$$y_n'(x) = y_0' + \int_{x_0}^{x} f_y(t, y_{n-1}'(t))dt$$

converges to a solution for the interval value problem

$y'' = y', x_0' = 0, y_0' = 0.$

we find that $y_0'(x) = 1$

$$y_1'(x) = 1 + \int_0^x (1 + t)dt = 1 + x + \frac{x^2}{2},$$

$$y_2'(x) = 1 + \int_0^x \left(1 + t + \frac{t^2}{2}\right) dt$$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

From the pattern that is developing, it is easy to conjecture that

$$y_n'(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$$

The limit of this sequence exists for every real number x because the limit is nothing more than the Maclaurin Series expansion for e^x, which converges for every x

That is,

$$\Phi'(x) = \lim_{n\to\infty} y_n'(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!} = e^x.$$

From uniqueness theorem

$Y'(x) = \lim_{n\to\infty} y_n'(x)$

From the above solution

$y_n'(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$

There fore

$Y'(x) = \lim_{n\to\infty} (\sum_{k=0}^{n} \frac{x^k}{k!}) = e^x$

$\therefore Y'(x) = e^x$

From the above solution $\emptyset'(x) = e^x$

Hence $Y'(x) = e^x = \Phi'(x)$

This proves the uniqueness of solution

Applications

A natural question now is the following:

How does one obtain useful information from a differential equation? The answer is essentially that if it is possible to do so, one shows the differential equation to obtain a solution; if this is not possible, one uses the theory of differential equation to obtain information about the solution.

The following are the some applications:

1. To determine the motion of a Projectile, Rocket, Satellite or Planet.
2. To determine the charge or current in an electric circuit.
3. To determine the vibrations of a wire or a membrane.
4. To study the rate of decomposition of a radioactive substance or the rate of growth of a population.
5. To study the reactions of chemicals.
6. To determine the curves that have certain geometrical properties.

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