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KRONECKER PRODUCT OF CONJUGATE UNITARY MATRICES

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ABSTRACT

The concept of Kronecker product of conjugate unitary matrices is introduced. Characterizations of Kronecker product of conjugate unitary matrices are obtained and derived some theorems.

INTRODUCTION

The first documented work on Kronecker products was written by Johann Georg Zehfuss between 1858 and 1868. He established the determinant result $|A \otimes B| = |A|^b|B|^a$, where A and B are square matrices of dimension a and b, respectively.
Zehfuss was acknowledged by Muir (1881) and his followers, who called the determinant $|A \otimes B|$ the Zehfuss determinant of A and B. In 1880's Kronecker gave a series of lectures in Berlin, where he introduced the result to his students [5]. In 1890's Harwitz and stephanos developed the determinant equality and other results involving Kronecker products.

The Kronecker product has a rich and very pleasing algebra that supports a wide range of fast, elegant and practical algorithms. Several trends in scientific computing suggest that this important matrix operation will have an increasingly greater role to play in the future [9].

The algebra of the Kronecker products of matrices is recapitulated using a notation that reveals the tensor structures of the matrices [7]. The generalized matrix product allows a wide family of unitary matrices to be developed from a single recursion formula [6].
Anna Lee has initiated the study of secondary symmetric matrices. Also she has shown that for a complex matrix A, the usual transpose A^T and secondary transpose A^s are related as $A^s = VA^TV$, where 'V' is the permutation matrix with units in its secondary diagonal.

Notations: Let $C_{n\times n}$ be the space of nxn complex matrices of order n. For $A \in C_{nxn}$. Let $A^T, \bar{A}$, A^*, A^s denote transpose, conjugate, conjugate transpose, secondary transpose of a matrix A respectively. Also the permutation matrix Vsatisfies $V^T = \bar{V} = V^* = V$and$V^2 = I$

A matrix $A \in C_{n\times n}$ is called unitary if $AA^* = A^*A = I$ [8]
A matrix $A \in C_{n\times n}$ is called normal if $AA^* = A^*A$ [3]
A matrix $A \in C_{n\times n}$ is called conjugate normal if $AA^* = \overline{A^*A}$ [1]
A matrix $A \in C_{n\times n}$ is called is called conjugate unitary if $AA^* = \overline{A^*A} = I$ [2]

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Matrices $A, B \in C_{n\times n}$ then $A\otimes B$ is called Kronecker product of matrix [4]

Kronecker Product of Conjugate Unitary Matrices

Definition: Given two matrices $A, B \in C_{n\times n}$ then the Kronecker Product

(tensor product or direct product) is defined as the matrix

$$A\otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \dots & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{bmatrix} \in C_{n\times n}$$

Theorem: If $A, B \in C_{n\times n}$ are conjugate unitary matrices then the Kronecker product $A\otimes B$ is conjugate unitary matrix.

Proof: Let $A, B \in C_{n\times n}$ are conjugate unitary matrices.

$\Longrightarrow AA^* = \overline{A^*A} = I$ and $BB^* = \overline{B^*B} = I$

Case (i) $(A\otimes B)(A\otimes B)^* = (A\otimes B)(A^*\otimes B^*)$

$= (A\,A^*)\otimes(BB^*)$

$= I\otimes I$ $(\because AA^* = \overline{A^*A} = I$ and $BB^* = \overline{B^*B} = I)$

$= I$

Case (ii) $\overline{(A\otimes B)^*(A\otimes B)} = \overline{(A^*\otimes B^*)(A\otimes B)}$

$= \overline{(A^*A)\otimes(B^*B)}$

$= \overline{I\otimes I}$ $(\because AA^* = \overline{A^*A} = I$ and $BB^* = \overline{B^*B} = I)$

$= \bar{I}$ $(\because \bar{I} = I)$

$= I$

Therefore, in both cases, we have $(A\otimes B)(A\otimes B)^* = \overline{(A\otimes B)^*(A\otimes B)} = I$

$\Longrightarrow (A\otimes B)$ is conjugate unitary matrix.

Theorem: If A$\in C_{n\times n}$ is a conjugate unitary matrices and I_n be identity matrix then $I_n\otimes A$ =diag$[A, A, A, \cdots A]$

Proof: Let A$\in C_{n\times n}$ be a conjugate unitary matrix and I_n be the identity matrix then

$$I_n\otimes A = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}\otimes A$$

$$= \begin{bmatrix} 1.A & \cdots & 0.A \\ \vdots & \ddots & \vdots \\ 0.A & \cdots & 1.A \end{bmatrix} \text{ (by using kronecker product)}$$

$$= \begin{bmatrix} A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A \end{bmatrix}$$

=diag$[A, A, A, \cdots A]$

Theorem: If $A, B \in C_{n\times n}$ are conjugate unitary matrices then the Kronecker product $(A^*\otimes B^*)$ is conjugate unitary matrix.

Proof: Let $A, B \in C_{n\times n}$ are conjugate unitary matrices.

$\Longrightarrow AA^* = \overline{A^*A} = I$ and $BB^* = \overline{B^*B} = I$

Case (i) $(A^*\otimes B^*)(A^*\otimes B^*)^* = (A^*\otimes B^*)((A^*)^*\otimes(B^*)^*)$

$= (A^*\otimes B^*)\,(A\otimes B)$ $(\because (A^*)^* = A$ and $(B^*)^* = B)$

$= (A^*A)\otimes(B^*B)$

$= I\otimes I$ $(\because AA^* = \overline{A^*A} = I$ and $BB^* = \overline{B^*B} = I)$

$= I$

Case (ii) $\overline{(A^*\otimes B^*)^*(A^*\otimes B^*)} = \overline{((A^*)^*\otimes(B^*)^*)(A^*\otimes B^*)}$

$= \overline{(A\otimes B)(A^*\otimes B^*)}$ $(\because (A^*)^* = A$ and $(B^*)^* = B)$

$= \overline{(A\,A^*)\otimes(BB^*)}$

$= \overline{I\otimes I}$ $(\because AA^* = \overline{A^*A} = I$ and $BB^* = \overline{B^*B} = I)$

$= \bar{I}$ $(\because \bar{I} = I)$

$= I$

Therefore, in both cases, we have $(A^* \otimes B^*)(A^* \otimes B^*)^* = \overline{(A^* \otimes B^*)^*(A^* \otimes B^*)} = I$
$\Longrightarrow (A^* \otimes B^*)$ is conjugate unitary matrix.

Theorem: If $A, B, C, D \in C_{n \times n}$ are conjugate unitary matrices, then the Kronecker product $((A \otimes B)(C \otimes D))$ is conjugate unitary matrix.

Proof: Let $A, B, C, D \in C_{n\,x\,n}$ are conjugate unitary matrices.
$\Longrightarrow AA^* = \overline{A^*A} = I, BB^* = \overline{B^*B} = I$, $CC^* = \overline{C^*C} = I$ and $DD^* = \overline{D^*D} = I$

Case (i) $((A \otimes B)(C \otimes D))((A \otimes B)(C \otimes D))^*$
$$\begin{aligned}
&= ((A \otimes B)(C \otimes D))((C \otimes D)^*(A \otimes B)^*)\\
&= ((A \otimes B)(C \otimes D))((C^* \otimes D^*)(A^* \otimes B^*))\\
&= ((A\,C) \otimes (BD))((C^*A^*) \otimes (D^*B^*))\\
&= ((A\,C)(C^*A^*)) \otimes ((BD)(D^*B^*))\\
&= (A(CC^*)A^*) \otimes (B(DD^*)B^*)\\
&= (AA^*) \otimes (BB^*) \ (\because CC^* = \overline{C^*C} = \mathrm{I} \ \& DD^* = \overline{D^*D} = I)\\
&= I \otimes I \ \ (\because AA^* = \overline{A^*A} = I \ \& \ BB^* = \overline{B^*B} = I)\\
&= I
\end{aligned}$$

Case (ii) $\overline{((A \otimes B)(C \otimes D))^*((A \otimes B)(C \otimes D))}$
$$\begin{aligned}
&= \overline{((C \otimes D)^*(A \otimes B)^*)((A \otimes B)(C \otimes D))}\\
&= \overline{((C^* \otimes D^*)(A^* \otimes B^*))((A \otimes B)(C \otimes D))}\\
&= \overline{((C^*A^*) \otimes (D^*B^*))((A\,C) \otimes (BD))}\\
&= \overline{((C^*A^*)(A\,C)) \otimes ((D^*B^*)(BD))}\\
&= \overline{(C^*(A^*A)C) \otimes (D^*(B^*B)D)}\\
&= \overline{(C^*C) \otimes (D^*D)} \ (\because AA^* = \overline{A^*A} = I \ \& BB^* = \overline{B^*B} = I)\\
&= I \otimes I \ \ \ (\because CC^* = \overline{C^*C} = \mathrm{I} \ \& \ DD^* = \overline{D^*D} = I)\\
&= I
\end{aligned}$$

Therefore, in both cases, we have $((A \otimes B)(C \otimes D))((A \otimes B)(C \otimes D))^* = \overline{((A \otimes B)(C \otimes D))^*((A \otimes B)(C \otimes D))} = I$
$\Longrightarrow ((A \otimes B)(C \otimes D))$ is conjugate unitary matrix.

Theorem: If $A, B \in C_{n \times n}$ are conjugate unitary matrices, then the Kronecker product $(A^s \otimes B^s)$ is conjugate unitary matrix.

Proof: Let $A, B \in C_{n \times n}$ are conjugate unitary matrices.

$\Longrightarrow AA^* = \overline{A^*A} = I$ and $BB^* = \overline{B^*B} = I$

Case (i) $(A^s \otimes B^s)((A^s \otimes B^s))^*$
$$\begin{aligned}
&= (A^s \otimes B^s)((A^s)^* \otimes (B^s)^*)\\
&= (A^s \otimes B^s)((A^*)^s \otimes (B^*)^s)\\
&= (A^s(A^*)^s \otimes B^s(B^*)^s)\\
&= ((A^*A)^s \otimes (B^*B)^s)\\
&= (I^s \otimes I^s) \quad (\because AA^* = \overline{A^*A} = I \text{ and } BB^* = \overline{B^*B} = I)\\
&= (I \otimes I) \qquad (\because I^s = I)\\
&= I
\end{aligned}$$

Case (ii) $\overline{((A^s \otimes B^s))^*(A^s \otimes B^s)}$
$$\begin{aligned}
&= \overline{((A^s)^* \otimes (B^s)^*)(A^s \otimes B^s)}\\
&= \overline{((A^*)^s \otimes (B^*)^s)(A^s \otimes B^s)}\\
&= \overline{((A^*)^sA^s \otimes (B^*)^sB^s)}\\
&= \overline{((AA^*)^s \otimes (BB^*)^s)}\\
&= \overline{(I^s \otimes I^s)} \qquad (\because AA^* = \overline{A^*A} = I \text{ and } BB^* = \overline{B^*B} = I)\\
&= \overline{(I \otimes I)} \qquad (\because I^s = I)\\
&= \bar{I}\\
&= I \qquad\qquad (\because \bar{I} = I)
\end{aligned}$$

Therefore, in both cases, we have $(A^s \otimes B^s)((A^s \otimes B^s))^* = \overline{((A^s \otimes B^s))^*(A^s \otimes B^s)} = I$
$\Longrightarrow (A^s \otimes B^s)$ is conjugate unitary matrix.

Theorem: If $A, B \in C_{n \times n}$ are conjugate unitary matrices and V is a permutation matrix then the Kronecker product $(VA \otimes VB)$ is conjugate unitary matrix.

Proof: Let $A, B \in C_{n \times n}$ are conjugate unitary matrices.
$\Longrightarrow AA^* = \overline{A^*A} = I$ and $BB^* = \overline{B^*B} = I$

Case (i) $(VA \otimes VB)((VA \otimes VB))^* = (VA \otimes VB)((VA)^* \otimes (VB)^*)$

$= (VA \otimes VB)(A^*V^* \otimes B^*V^*)$
$= (VA \otimes VB)(A^*V \otimes B^*V) \quad (\because V^* = V)$
$= \big((VA)(A^*V)\big) \otimes \big((VB)(B^*V)\big)$
$= (V(AA^*)V) \otimes (V(BB^*)V)$
$= (VV) \otimes (VV) \quad (\because AA^* = \overline{A^*A} = I \text{ and } BB^* = \overline{B^*B} = I)$
$= V^2 \otimes V^2$
$= I \otimes I \quad (\because V^2 = I)$
$= I$

Case (ii) $\overline{\big((VA \otimes VB)\big)^*(VA \otimes VB)} = \overline{\big((VA)^* \otimes (VB)^*(VA \otimes VB)\big)}$
$= \overline{\big(((VA)^*VA) \otimes ((VB)^*VB)\big)}$
$= \overline{\big(((A^*V^*)VA) \otimes ((B^*V^*)VB)\big)}$
$= \overline{\big((A^*(V^*V)A) \otimes (B^*(V^*V)B\big)}$
$= \overline{\big((A^*(VV)A) \otimes (B^*(VV)B\big)} \quad (\because V^* = I)$
$= \overline{\big((A^*V^2A) \otimes (B^*V^2B\big)}$
$= \overline{\big((A^*A) \otimes (B^*B\big)} \quad (\because V^2 = I)$
$= \overline{(I \otimes I)} \quad (\because AA^* = \overline{A^*A} = I \text{ and } BB^* = \overline{B^*B} = I)$
$= \bar{I}$
$= I \quad (\because \bar{I} = I)$

Therefore, in both cases, we have $(VA \otimes VB)\big((VA \otimes VB)\big)^* = \overline{\big((VA \otimes VB)\big)^*(VA \otimes VB)} = I$
$\Longrightarrow (VA \otimes VB)$ is conjugate unitary matrix.

Theorem: Let A, B, C and $D \in C_{n\times n}$ are conjugate unitary matrices then $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

Proof:
$$(A \otimes B)(C \otimes D) = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \cdots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}\begin{bmatrix} c_{11}D & \cdots & c_{1n}D \\ \vdots & \cdots & \vdots \\ c_{n1}D & \cdots & c_{nn}D \end{bmatrix}$$
$$= \begin{bmatrix} \sum_{k=1}^{n}[a_{1k}c_{k1}BD] & \cdots & \sum_{k=1}^{n}[a_{1k}c_{kn}BD] \\ \vdots & \cdots & \vdots \\ \sum_{k=1}^{n}[a_{nk}c_{k1}BD] & \cdots & \sum_{k=1}^{n}[a_{nk}c_{kn}BD] \end{bmatrix}$$
$$= \begin{bmatrix} \sum_{k=1}^{n}[a_{1k}c_{k1}] & \cdots & \sum_{k=1}^{n}[a_{1k}c_{kn}] \\ \vdots & \cdots & \vdots \\ \sum_{k=1}^{n}[a_{nk}c_{k1}] & \cdots & \sum_{k=1}^{n}[a_{nk}c_{kn}] \end{bmatrix} \otimes BD$$
$\therefore (A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

Theorem: If $A_1, A_2, \cdots, A_n$ and $B_1, B_2, \cdots, B_n$ are sequence of conjugate unitary matrices in $C_{n\times n}$, then $(A_1 \otimes B_1)(A_2 \otimes B_2)\cdots(A_n \otimes B_n) = (A_1A_2\cdots A_n) \otimes (B_1B_2\cdots B_n)$.

Theorem: If $A_1, A_2, \cdots, A_m$ and $B_1, B_2, \cdots, B_m$ are sequence of conjugate unitary matrices in $C_{n\times n}$, then $(A_1 \otimes A_2 \otimes \cdots \otimes A_m)(B_1 \otimes B_2 \otimes \cdots \otimes B_m) = (A_1B_1) \otimes (A_2B_2) \otimes \cdots \otimes (A_mB_m)$

Theorem: If A and B are conjugate unitary matrices in $C_{n\times n}$, then $(A \otimes B)$ is conjugate normal.

Proof: Given that A and B are conjugate unitary matrices in $C_{n\times n}$.
$\Longrightarrow$ A and B are conjugate normal matrices.

That is $AA^* = \overline{A^*A}$ and $BB^* = \overline{B^*B}$

We have show that $(A \otimes B)(A \otimes B)^* = \overline{(A \otimes B)^*(A \otimes B)}$
Case (i) $(A \otimes B)(A \otimes B)^* = (A \otimes B)(A^* \otimes B^*)$
$= (A\,A^*) \otimes (BB^*)$
Case (ii) $\overline{(A \otimes B)^*(A \otimes B)} = \overline{(A^* \otimes B^*)(A \otimes B)}$
$= \overline{(A^*A) \otimes (B^*B)}$
$= (\overline{A^*A}) \otimes (\overline{B^*B})$
$= (AA^*) \otimes (BB^*) \quad (\because AA^* = \overline{A^*A} \text{ and } BB^* = \overline{B^*B})$

Therefore, in both cases, we have $(A \otimes B)(A \otimes B)^* = \overline{(A \otimes B)^*(A \otimes B)}$
$\Longrightarrow (A \otimes B)$ is conjugate normal matrix.

CONCLUSION

The Kronecker product of conjugate unitary matrix was defined and theorems related Kronecker product of conjugate unitary matrices are derived. This concept may be applied to secondary transpose and conjugate transpose of matrices.

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