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Research Article

GENERALIZED INVEXITY AND INVARIANT MONOTONICITY

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ABSTRACT

In this paper, several kinds of invariant monotone maps and generalized invariant monotone maps are introduced. Some examples are given which show that invariant monotonicity and generalized invariant monotonicity are proper generalizations of monotonicity and generalized monotonicity. Relationships between generalized invarant monotonicity and generalized invexity are established. Our results are generalizations of those presented by Pradhan S.K., and D.K. Dalai.

INTRODUCTION

Convexity is a common assumption made in Mathematical programming. In recent years, there have been increasing attempts to weaken the convexity conditions. Consequently, several classes of (generalized) invex functions have been introduced in the literature. More specifically, the concept of invexity was introduced by Hanson [2], where it is shown that the Kuhn-Tucker conditions are sufficient for (global) optimality of non-linear programming problems under invexity condition. Weir and Mond [5] - [6] introduced the concept of pre invex functions and applied it to the establishment of the sufficient optimality conditions and duality in (multi objective) non-linear programming. Mohan and Neogy [3] shown that, under certain conditions an invex function is preinvex.

In this paper we further generalized the idea of X.M. Yang, X.M.Yang and K.L. Teo [7] taking into account of four variables instead of three variables. We introduced several types of generalized invariant monotonicities which are generalization of the (strict) monotonicity. The main aim of this chapter is to establish relations among generalized invariant monotonicities and generalized invexities.

Invariant Monotone Maps And Strictly Invariant Monotone Maps

Let Γ be non-empty subset of R^n. Let η be a vector valued function from $X \times X \times X \times X$ into R^n ($X \subset R^n$) and Let F be a vector function from Γ into $R^n \times R^n \times R^n$.

Definition: A set Γ is said to be invex with respect to η if there exists an

$\eta : R^n \times R^n \times R^n \times R^n \to R^n$ such that for any p, q, r, s $\in \Gamma$ and $\lambda \in [0, 1]$,
$p + \lambda\eta (p, q, r, s) \in \Gamma$
$q + \lambda\eta (s, p, q, r) \in \Gamma$

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$r + \lambda\eta\ (q, r, s, p) \in \Gamma$
$s + \lambda\eta\ (r, s, p, q) \in \Gamma$

Definition

F is said to be Monotone on Γ if for every pair of points p, q, r, s $\in \Gamma$,
$(q - p)^T\ (F(q) - F(p)) \geq 0.$

Definition

Let Γ be an invex set with respect to η. F is said to be invariant Monotone on Γ with respect to η if for any p, q, r, s $\in \Gamma$,
$\eta(p, q, r, s)^T F(s) + \eta(s, p, q, r)^T F(r) + \eta(r, s, p, q)^T F(q) + \eta(q, r, s, p)^T F(p) \leq 0.$

Remark

Every Monotone map is an invariant Monotone Map with

$\eta(p, q, r, s) = 2s - p - r,$
$\eta(s, p, q, r) = 2r - q - s$
$\eta(r, s, p, q) = 2q - p - r,$
$\eta(q, r, s, p) = 2p - q - s$

F is monotone on Γ, we have for every pair of points p, q, r, s $\in \Gamma$,
$(s - r)^T F(s) + (r - q)^T F(r) + (q - p)^T F(q) + (p - s)^T F(p) \geq 0$
$\Rightarrow [F(s) - F(r)]\ (s - r) + [F(r) - F(q)]\ (r - q) + [F(q) - F(p)]\ (q - p) + [F(p) - F(s)](p - s) \leq 0$
$\Rightarrow (2s - p - r)\ F(s) + (2r - s - q)\ F(r) + (2q - r - p)\ F(q) + (2p - q - s)\ F(p) \leq 0$
$\Rightarrow \eta(p, q, r, s)^T F(s) + \eta(s, p, q, r)^T F(r) + \eta(r, s, p, q)^T F(q) + \eta(q, r, s, p)^T F(p) \leq 0$

F is invariant Monotone with respect to η on Γ, but the converse is not true.

Example : Let F and η be Maps defined as

$F(p) = (1 + \cos p_1, 1 + \cos p_2, 1 + \cos p_3, 1 + \cos p_4)$
$F(q) = (1 + \cos q_1, 1 + \cos q_2, 1 + \cos q_3, 1 + \cos q_4)$
$F(r) = (1 + \cos r_1, 1 + \cos r_2, 1 + \cos r_3, 1 + \cos r_4)$
$F(s) = (1 + \cos s_1, 1 + \cos s_2, 1 + \cos s_3, 1 + \cos s_4)$

$$\eta(p,\ q,\ r,\ s) = \Bigg[(1+\cos q_1)(\sin p_1 - \sin q_1)\frac{\sin s_1}{\cos q_1}, (1+\cos q_2)(\sin p_2 - \sin q_2)\frac{\sin s_2}{\cos q_2},$$
$$(1+\cos q_3)(\sin p_3 - \sin q_3)\frac{\sin s_3}{\cos q_3}, (1+\cos q_4)\,(\sin p_4 - \sin q_4)\frac{\sin s_4}{\cos q_4} \Bigg]$$

where p, q, r, s . $\in \left(0,\ \frac{\pi}{2}\right)\times\left(0,\ \frac{\pi}{2}\right)\times\left(0,\ \frac{\pi}{2}\right)\times\left(0,\ \frac{\pi}{2}\right)$

$\eta(p, q, r, s)^T F(s) + \eta(s, p, q, r)^T F(r) + \eta(r, s, p, q)^T F(q) + \eta(q, r, s, p)^T F(p)$

$$= \Bigg[(1+\cos r_1)(\sin p_1 - \sin q_1)\frac{\sin r_1}{\cos r_1} + (1+\cos r_2)(\sin p_2 - \sin q_2)\frac{\sin r_2}{\cos r_2}$$
$$+ (1+\cos r_3)(\sin p_3 - \sin q_3)\frac{\sin r_3}{\cos r_3} + (1+\cos r_4)(\sin p_4 - \sin q_4)\frac{\sin r_4}{\cos r_4} \Bigg]$$
$$+ \Bigg[(1+\cos q_1)(\sin r_1 - \sin p_1)\frac{\sin q_1}{\cos q_1} + (1+\cos q_2)(\sin r_2 - \sin p_2)\frac{\sin q_2}{\cos q_2}$$
$$+ (1+\cos q_3)(\sin r_3 - \sin p_3)\frac{\sin q_3}{\cos q_3} + (1+\cos q_4)(\sin r_4 - \sin p_4)\frac{\sin q_4}{\cos q_4} \Bigg]$$

$$+\left[(1+\cos p_1)(\sin q_1-\sin r_1)\frac{\sin p_1}{\cos p_1}+(1+\cos p_2)(\sin q_2-\sin r_2)\frac{\sin p_2}{\cos p_2}\right.$$

$$\left.+(1+\cos p_3)(\sin q_3-\sin r_3)\frac{\sin p_3}{\cos p_3}+(1+\cos p_4)(\sin q_4-\sin r_4)\frac{\sin p_4}{\cos p_4}\right]$$

$$=\sum\left[\frac{\sin p_1\sin r_1-\sin q_1\sin r_1}{\cos r_1}+\frac{\sin r_1\sin q_1-\sin p_1\sin q_1}{\cos q_1}+\frac{\sin p_1\sin q_1-\sin p_1\sin r_1}{\cos p_1}\right]$$

$$=\frac{\sum\{(\sin p_1\sin r_1-\sin q_1\sin r_1)\cos p_1\cos q_1+(\sin r_1\sin q_1-\sin p_1\sin q_1)\cos p_1\cos r_1+(\sin p_1\sin q_1-\sin p_1\sin r_1)\cos q_1\cos r_1\}}{\cos p_1\cos q_1\cos r_1}$$

$$=\frac{\sum[\sin p_1\sin r_1\cos q_1(\cos p_1-\cos r_1)+\sin r_1\sin q_1\cos p_1(\cos q_1-\cos r_1)+(\sin p_1\sin q_1\cos r_1(\cos q_1-\cos p_1)]}{\cos p_1\cos q_1\cos r_1}$$

$$=\sum\{\tan p_1\tan r_1(\cos p_1-\cos r_1)+\tan q_1\tan r_1(\cos q_1-\cos r_1)+\tan p_1\tan q_1(\cos q_1-\cos p_1)\}$$

$$=\sum\left[-2\left\{\tan p_1\tan r_1\sin\left(\frac{p_1+r_1}{2}\right)\sin\left(\frac{p_1-r_1}{2}\right)+\tan q_1\tan r_1\sin\left(\frac{q_1+r_1}{2}\right)\sin\left(\frac{q_1-r_1}{2}\right)\right.\right.$$

$$\left.\left.+\tan p_1\tan q_1\sin\left(\frac{q_1+p_1}{2}\right)\sin\left(\frac{q_1-p_1}{2}\right)\right\}\right]$$

for p, q, r, s $\in \left(0,\frac{\pi}{2}\right)\times\left(0,\frac{\pi}{2}\right)\times\left(0,\frac{\pi}{2}\right)\times\left(0,\frac{\pi}{2}\right)$

Hence F is invariant monotone with respect to η.

Let $p=\left(\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{4}\right)$, $q=\left(\frac{\pi}{6},\frac{\pi}{6},\frac{\pi}{6},\frac{\pi}{6}\right)$

Now, $(q - p)^T[F(q) - F(p)]$

$$=\left(\frac{\pi}{6}-\frac{\pi}{4}\right)\left[\left(1+\cos\frac{\pi}{6},1+\cos\frac{\pi}{6},1+\cos\frac{\pi}{6},1+\cos\frac{\pi}{6}\right)-\left(1+\cos\frac{\pi}{4},1+\cos\frac{\pi}{4},1+\cos\frac{\pi}{4},1+\cos\frac{\pi}{4}\right)\right]$$

$$=\left(-\frac{\pi}{12}\right)\sqrt{4\left(\cos\frac{\pi}{6}-\cos\frac{\pi}{4}\right)^2}$$

$$=\left(-\frac{\pi}{6}\right)\left(\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\right)<0$$

Thus F is not Monotone.

Definition

Let the set Γ be invex with respect to η and $\eta : X \times X \times X \times X \to R^n$ be a vector valued function. The function $f : \Gamma \to R$ is said to be pre-invex with respect to η if

$f(p + \lambda\eta(p, q, r, s) \leq \lambda f(p) + \lambda_1 f(q) + \lambda_2 f(r) + \lambda_3 f(s)$

$f(q + \lambda\eta(s, p, q, r) \leq \lambda f(s) + \lambda_1 f(p) + \lambda_2 f(q) + \lambda_3 f(r)$

$f(r + \lambda\eta(q, r, s, p) \leq \lambda f(q) + \lambda_1 f(r) + \lambda_2 f(s) + \lambda_3 f(p)$

$f(s + \eta\lambda(r, s, p, q) \leq \lambda f(s) + \lambda_1 f(s) + \lambda_2 f(p) + \lambda_3 f(q)$

$\forall$ p, q, r, s $\in \Gamma$ where $\lambda + \lambda_1 + \lambda_2 + \lambda_3 = 1$; $\lambda \in [0, 1]$.

Assumption - A

Let the set Γ be convex with respect to η and let $f : \Gamma \to R$. Then
$f(s + \lambda\eta(p, q, r, s)) \leq f(p)$
$f(r + \lambda\eta(s, p, q, r)) \leq f(q)$
$f(q + \lambda\eta(q, r, s, p)) \leq f(r)$
$f(p + \lambda\eta(r, s, p, q)) \leq f(s)$

Remark

Assumption A is just the inequality of the definition of pre-invexity with
$\lambda = 1, \lambda_1 = \lambda_2 = \lambda_3 = 0$

Assumption - B

Let $\eta : X \times X \times X \times X \to R^n$. Then for any $p, q, r, s \in R^n$ and for $\lambda \in [0, 1]$,
$\eta(q, q, q, q + \lambda\eta(p, q, r, s)) = -\lambda\eta(p, q, r, s)$
$\eta(p, q + \lambda_1\eta, r, s) = (1 - \lambda_1)\eta\ (p, q, r, s)$
$\eta(p, q, r + \lambda_2\eta, s) = (1 - \lambda_2)\eta\ (p, q, r, s)$
$\eta(p, q, r, s + \lambda_3\eta) = (1 - \lambda_3)\ \eta(p, q, r, s),$
where $\lambda + \lambda_1 + \lambda_2 + \lambda_3 = 1$.

Remark: We will show that Assumption B holds if
$\eta(p, q, r, s) = (3p - q - r - s) + o\|3p - q - r - s\|$

(a) $\eta(q, q, q, q + \lambda\ \eta(p, q, r, s))$
$= \eta(q, q, q, q + \lambda[(3p - q - r - s) + o\|3p - q - r - s\|)$
$= 3q - q - q - q - (q + \lambda)\ [(3p - q - r - s) + 0\|3p - q - r - s\|)$
$= -\lambda((3p - q - r - s) + o\|3p - q - r - s)\|)$
$= -\lambda\eta(p, q, r, s)$

(b) $\eta(p, q + \lambda_1\eta, r, s)$
$= (3p - q - \lambda_1\eta - r - s) + o\|3p - q - \lambda_1\eta - r - s\|$
$= (3p - q - r - s) - \lambda_1\eta(p, q, r, s) + o\|(3p - q - r - s) - \lambda_1\eta(p, q, r, s)\|$
$= (3p - q - r - s) - \lambda_1(3p - q - r - s) + o\|3p - q - r - s\| + o\|(3p + q - r - s) - \lambda_1(3p - q - r - s) + o\|3p - q - r - s)\|$
$= (1 - \lambda_1)\ (3p - q - r - s) + (1 - \lambda_1)\ o\|3p - q - r - s\|$
$= (1 - \lambda_1)\ ((3p - q - r - s) + o\|3p - q - r - s\|)$
$= (1 - \lambda_1)\ \eta\ (p, q, r, s)$

(c) $\eta(p, q, r + \lambda_2\eta, s)$
$= (3p - q - r - \lambda_2\eta - s) + o\|3p - q - r - \lambda_2\eta - s\|$
$= (3p - q - r - s) - \lambda_2 n(p, q, r,s) + o\|(3p - q - r - s) - \lambda_2\eta(p, q, r, s)\|$
$= (3p - q - r - s) - \lambda_2((3p - q - r - s) + o\|3p - q - r - s\|) + o\|(3p - q - r - s) - \lambda_2((3p - q - r - s) + o\|3p - q - r - s)\|)$
$= (1 - \lambda_2)\ (3p - q - r - s) + (1 - \lambda_2)\ o\|3p - q - r - s\|$
$= (1 - \lambda_2)\ ((3p - q - r - s) + o\|3p - q - r - s\|)$
$= (1 - \lambda_2)\ \eta(p, q, r, s)$

(d) $\eta(p, q, r, s + \lambda_3\eta)$
$= (3p - q - r - s - \lambda_3\eta) + o\|3p - q - r - s - \lambda_3\eta)\|$
$= 3p - q - r - s - \lambda_3((3p - q - r - s) + o\|3p - q - r - s\|) + o\|((p - q - r - s) - \lambda_3((3p - q - r - s\|) + o\|3p - q - r - s)\|$
$= (1 - \lambda_3)\ (3p - q - r - s) + (1 + \lambda_3)\ o\|3p - q - r - s\|$
$= (1 - \lambda_3)\ ((3p - q - r - s) + o\|3p - q - r - s\|)$
$= (1 - \lambda_3)\ \eta(p, q, r, s)$

Example

Let $f(x) = -|x|, \forall\ x \in K = (-2, 2)$ and let

$$\eta(p, q, r, s) = \begin{cases} p+q+r-s, & p \geq 0, q \geq 0, r \geq 0, s \geq 0 \\ p+q+r-s, & p < 0, q \geq 0, r < 0, s < 0 \\ p+q+r-s, & p < 0, q < 0, r \geq 0, s < 0 \\ p+q+r-s, & p < 0, q < 0, r < 0, s \geq 0 \\ p+q+r-s, & p \geq 0, q \geq 0, r < 0, s < 0 \\ p+q+r-s, & p \geq 0, q < 0, r \geq 0, s < 0 \\ p+q+r-s, & p \geq 0, q < 0, r < 0, s \geq 0 \\ p+q+r-s, & p < 0, q < 0, r \geq 0, s \geq 0 \\ -2-(q+r+s), & p \geq 0, q \geq 0, r < 0, s < 0 \\ -2-(q+r+s), & p \geq 0, q \geq 0, r < 0, s \geq 0 \\ -2-(q+r+s), & p \geq 0, q < 0, r \geq 0, s \geq 0 \\ -2-(q+r+s), & p \geq 0, q < 0, r < 0, s < 0 \\ -2-(q+r+s), & p < 0, q < 0, r < 0, s \geq 0 \\ -2-(q+r+s), & p < 0, q \geq 0, r < 0, s \geq 0 \\ -2-(q+r+s), & p < 0, q \geq 0, r \geq 0, s \geq 0 \\ -2-(q+r+s), & p < 0, q < 0, r \geq 0, s < 0 \end{cases}$$

Thus, f is invex with respect to η on K and that f and η satisfy Assumptions A and B. However, f is not convex.

The following theorem shows that the preinverxity of a function is equivalent to the invariant monotone property of its gradient. This is a generalization of the convexity of a function and the monotonicity of its gradient obtained in [1].

Lemma

Let f be differentiable on an open set containing Γ. If f is invex with respect to η, then ∇f is an invariant monotone with respect to η.

Proof

Let f be invex. Then we have $f(p + \lambda\eta(p, q, r, s) \leq \lambda f(p) + \lambda_1 f(q) + \lambda_2 f(r) + \lambda_3 f(s)$,

where $p, q, r, s \in \Gamma$, $\lambda + \lambda_1 + \lambda_2 + \lambda_3 = 1$, $\lambda \in [0, 1]$

Applying Mean Value Theorem, $f(p + \lambda\eta) - f(p) = \lambda\eta\, f'(p + \theta\lambda\eta)$

$$\Rightarrow \frac{f(p+\lambda\eta)-f(p)}{\lambda\eta} = f'(p+\theta\lambda\eta) \tag{1}$$

Here, $f(p + \lambda\eta(p, q, r, s) - f(p) \leq \lambda f(p) - f(p) + \lambda_1 f(q) + \lambda_2 f(r) + \lambda_3 f(s))$

$$\Rightarrow \frac{f(p+\eta\lambda)(p,q,r,s)-f(p)}{\lambda\eta} \leq \frac{(\lambda-1)f(p)+\lambda_1 f(q)+\lambda_2 f(r)+\lambda_3 f(s)}{\lambda\eta}$$

From equation (1), we get

$$f'(p+\theta\lambda\eta) \leq \frac{(\lambda-1)f(p)+\lambda_1 f(q)+\lambda_2 f(r)+\lambda_3 f(s)}{\lambda\eta}$$

$\Rightarrow \lambda\eta^T \nabla f(p) \leq (\lambda - 1)f(p) + \lambda_1 f(q) + \lambda_2 f(r) + \lambda_3 f(s)$ (i)

Similarly,

$\lambda\eta^T \nabla f(q) \leq (\lambda_2 - 1) f(q) + \lambda f(s) + \lambda_1 f(p) + \lambda_3 f(r)$ (ii)

$\lambda\eta^T \nabla f(r) \leq (\lambda_1 - 1) f(r) + \lambda f(q) + \lambda_2 f(s) + \lambda_3 f(p)$ (iii)

$\lambda\eta^T \nabla f(s) \leq (\lambda_1 - 1) f(s) + \lambda f(r) + \lambda_2 f(p) + \lambda_3 f(q)$ (iv)

Adding (i), (ii), (iii) and (iv), we have

$\eta(p, q, r, s)^T \nabla f(s) + \eta(s, p, q, r)^T \nabla f(r) + \eta(q, r, s, p)^T \nabla f(q) + \eta(r, s, p, q)^T \nabla f(p)$

$\leq (\lambda - 1 + \lambda_3 + \lambda_2 + \lambda_1)$ f(p) + $(\lambda_2 - 1 + \lambda_1 + \lambda + \lambda_3)$ f(q) + $(\lambda_3 -1 + \lambda_2 + \lambda + \lambda_1)$ f(r)
+ $(\lambda - 1 + \lambda_3 + \lambda_1 + \lambda_2)$f(s) ≤ 0, for $\lambda + \lambda_1 + \lambda_2 + \lambda_3 = 1$, $\lambda \in [0, 1]$
∇f is invariant monotone on Γ with respect to η.

Theorem

Let f and η satisfy respectively Assumptions A and B and let f be differentiable on Γ. Then f is a preinvex function with respect to η on Γ if and only if ∇f is invariant monotone with respect to η on Γ and f satisfies Assumption A.

Proof

Suppose f is pre invex on Γ with respect to Γ Then
$f(p + \lambda\eta(p, q, r, s)) \leq \lambda f(p) + \lambda_1 f(q) + \lambda_2 f(r) + \lambda_3 f(s)$
$f(q + \lambda\eta(s, p, q, r) \leq f(s) + \lambda_1 f(p) + \lambda_2 f(q) + \lambda_3 f(r)$
$f(r + \lambda\eta(q, r, s, p) \leq \lambda f(q) + \lambda_1 f(r) + \lambda_2 f(s) + \lambda_3 f(p)$
$f(s + \lambda\eta(r, s, p, q) \leq \lambda f(r) + \lambda_1 f(s) + \lambda_2 (p) + \lambda_3 f(q)$
By assumption A, $\lambda = 1$, $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

The invexity of a function is equivalent to the invariant monotone property of its gradient by lemma 2.1.
Since f is differentiable on an open set cotaining Γ, therefore ∇f is invariant with respect to η on Γ. Conversely, suppose that ∇f is invariant monotone with respect to η on Γ. Assume that f is not preinvex with respect to η on Γ. Then there exist p, q, r, s $\in \Gamma$ such that

$f(p + \eta(p, q, r, s) > \lambda f(p) + \lambda_1 f(q) + \lambda_2 f(r) + \lambda_3 f(s)$, where p, q, r, s $\in \Gamma$,
$\lambda + \lambda_1 + \lambda_2 + \lambda_3 = 1$ and $\lambda \in [0, 1]$.

Applying Mean Value Theorem.
$f(p + \lambda\eta) - f(p) = \lambda\eta f'(p + \theta\lambda\eta)$, where $0 < \theta < 1$

$$\Rightarrow \frac{f(p+\lambda\eta)-f(p)}{\lambda\eta} = f'(p+\theta\lambda\eta)$$

Now, $f(p + \lambda\eta(p, q, r, s)) - f(p) > \lambda f(p) - f(p) + \lambda_1 f(q) + \lambda_2 f(r) + \lambda_3 f(s)$
$> (\lambda - 1) f(p) + \lambda_1 f(q) + \lambda_2 f(r) + \lambda_3 f(s)$

$$\Rightarrow \frac{f(p+\lambda\eta)(p,q,r,s)-f(p)}{\lambda\eta} > \frac{(\lambda-1)f(p)+\lambda_1 f(q)+\lambda_2 f(r)+\lambda_3 f(s)}{\lambda\eta}$$

Using equation (1), we get

$$f'(p+\theta\lambda\eta) > \frac{(\lambda-1)f(p)+\lambda_1 f(q)+\lambda_2 f(r)+\lambda_3 f(s)}{\lambda\eta}$$

$\Rightarrow \lambda\eta^T \nabla f(p) > (\lambda - 1)f(p) + \lambda_1 f(q) + \lambda_2 f(r) + \lambda_3 f(s)$ (i)
Similarly, we get
$\lambda\eta^T \nabla f(q) > (\lambda_2 - 1) f(q) + \lambda f(s) + \lambda_1 f(p) + \lambda_3 f(r)$ (ii)
$\lambda\eta^T \nabla f(r) > (\lambda_1 - 1) f(r) + \lambda f(q) + \lambda_2 f(s) + \lambda_3 f(p)$ (iii)
$\lambda\eta^T f(s) > (\lambda_1 - 1) f(s) + \lambda f(r) + \lambda_2 f(p) + \lambda_3 f(q)$ (iv)
Adding equation (i), (ii), (iii) and (iv) we have

$\eta(p, q, r, s)^T \nabla f(s) + \eta(s, p, q, r)^T \nabla f(r) + \eta(q, r, s, p)^T \nabla f(q) + \eta(r, s, p, q)^T \nabla f(p) >$
$(\lambda - 1 + \lambda_1 + \lambda_2 + \lambda_3)f(p) + (\lambda_2 -1 + \lambda_1 + \lambda + \lambda_3)f(q) + (\lambda_1 - 1 + \lambda_2 + \lambda_3 + \lambda)$ f(r) $+(\lambda_1 - 1 + \lambda_2 + \lambda + \lambda_3)f(s) = 0$,
which contradicts the invariant monotonicity with respect to η on Γ.

Definition

F is said to be stricity monotone on Γ if for every pair of distinct point (p, q) $\in \Gamma$,
$(q - p)^T(F(q) - F(p)) > 0$

Definition

Let Γ be an invex set with respect to η. F is said to be strictly invariant monotone with respect to η on Γ if for any point p, q, r, s $\in \Gamma$,

$\eta(p, q, r, s)^T \nabla f(s) + \eta(s, p, q, r)^T \nabla f(r) + \eta(q, r, s, p)^T \nabla f(p) + \eta(r, s, p, q)^T \nabla f(q) \le 0$

Remark

Every monotone map is an invariant monotone map with

$\eta(p, q, r, s) = 2s - p - r$, $\quad \eta(q, r, s, p) = 2r - q - s$

$\eta(s, p, q, r) = 2p - q - s$, $\quad \eta(r, s, p, q) = 2q - p - r$.

F is monotone on Γ.

We have for every pair of points p, q, r, s $\in \Gamma$.

$(s - r)^T F(s) + (r - q)^T F(r) + (q - p)^T F(q) + (p - s)^T F(p) > 0$

$\Rightarrow (F(s) - F(r))\,(s - r) + (F(r) - F(q))\,(r - q) + (F(q) - F(p))\,(q - p) + (F(p) - F(s))\,(p - s) \le 0$

$\Rightarrow (2s - p - r)\,F(s) + (2r - s - q)\,F(r) + (2q - r - p)F(q) + (2p - q - s)\,F(p) \le 0$

$\Rightarrow \eta(p, q, r, s)^T F(s) + \eta(s, p, q, r)^T \cdot F(r) + \eta(q, r, s, p)^T F(q) + \eta(r, s, p, q)^T F(p) \le 0$

F is invariant monotone with respect to η on Γ but the converse is not true.

Example

Define the map Γ and η as

$F(p) = (-1 - \cos p_1, -1 - \cos p_2, - 1 - \cos p_3, -1 - \cos p_4)$

$F(q) = (-1 - \cos q_1, - 1 - \cos q_2, -1 - \cos q_3, -1 - \cos q_4)$

$F(r) = (-1 - \cos r_1, -1 - \cos r_2, -1 - \cos r_3, -1 - \cos r_4)$

$F(s) = (-1 - \cos s_1, -1 - \cos s_2, -1 - \cos s_3, -1 - \cos s_4)$

$$\eta(p, q, r, s) = \Bigg[(-1-\cos q_1)(\sin p_1 - \sin q_1)\frac{\sin s_1}{\cos q_1}, (-1-\cos q_2)(\sin p_2 - \sin q_2)\cdot\frac{\sin s_2}{\cos q_2},$$

$$(-1-\cos q_3)(\sin p_3 - \sin q_3)\frac{\sin s_3}{\cos q_3}, (-1-\cos q_4)(\sin p_4 - \sin q_4)\cdot\frac{\sin s_4}{\cos q_4}\Bigg]$$

where p, q, r, s $\in \left(0, \frac{\pi}{2}\right)\times\left(0, \frac{\pi}{2}\right)\times\left(0, \frac{\pi}{2}\right)\times\left(0, \frac{\pi}{2}\right)$

$\eta(p, q, r, s)^T F(s) + \eta(s, p, q, r)^T F(r) + \eta(q, r, s, p)^T F(q) + \eta(r, s, p, q)^T F(p)$

$$= \Bigg[(-1-\sin s_1)\left(\frac{\cos q_1 - \cos p_1}{\sin s_1}\right) + (-1-\sin s_2)\left(\frac{\cos q_2 - \cos p_2}{\sin s_2}\right)$$

$$+(-1-\sin s_3)\left(\frac{\cos q_3 - \cos p_3}{\sin s_3}\right) + (1-\sin s_4)\left(\frac{\cos q_4 - \cos p_4}{\sin s_4}\right)\Bigg]$$

$$+\Bigg[(-1-\sin q_1)\left(\frac{\cos p_1 - \cos s_1}{\sin q_1}\right) + (-1-\sin q_2)\left(\frac{\cos p_2 - \cos s_2}{\sin q_2}\right)$$

$$+(-1-\sin q_3)\left(\frac{\cos p_3 - \cos s_3}{\sin q_3}\right) + (1-\sin q_4)\left(\frac{\cos p_4 - \cos s_4}{\sin q_4}\right)\Bigg]$$

$$+\Bigg[\left(-1-\sin p_1\right)\left(\frac{\cos s_1 - \cos q_1}{\sin p_1}\right) + \left(-1-\sin p_2\right)\left(\frac{\cos s_2 - \cos q_2}{\sin p_2}\right)\Bigg]$$

$$+(-1-\sin p_3)\left(\frac{\cos s_3 - \cos q_3}{\sin p_3}\right) + (-1-\sin p_4)\left(\frac{\cos s_4 - \cos p_4}{\sin p_4}\right)\Bigg]$$

$$+\left[(-1-\sin r_1)\left(\frac{\cos s_1-\cos p_1}{\sin r_1}\right)+(-1-\sin r_2)\left(\frac{\cos s_2-\cos p_2}{\sin r_2}\right)\right.$$

$$\left.+(-1-\sin r_3)\left(\frac{\cos s_3-\cos p_3}{\sin r_3}\right)+(-1-\sin r_4)\left(\frac{\cos s_4-\cos p_4}{\sin r_4}\right)\right]$$

$$=\sum(-1-\sin s_1)\left(\frac{\cos q_1-\cos p_1}{\sin s_1}\right)+(-1-\sin q_1)\left(\frac{\cos p_1-\cos s_1}{\sin q_1}\right)$$

$$+(-1-\sin p_1)\frac{\cos s_1-\cos q_1}{\sin p_1}+(-1-\sin r_1)\left(\frac{\cos s_1-\cos p_1}{\sin r_1}\right)$$

$$=\sum\frac{\cos p_1-\cos q_1}{\sin s_1}+\frac{\cos s_1-\cos p_1}{\sin q_1}+\frac{\cos q_1-\cos s_1}{\sin p_1}+\frac{\cos p_1-\cos s_1}{\sin r_1}$$

$$=\sum\frac{-2}{\sin s_1}\left(\sin\frac{p_1+q_1}{2}\sin\frac{p_1-q_1}{2}\right)+\left(\frac{-2}{\sin q_1}\right)\left(\sin\frac{s_1+p_1}{2}\sin\frac{s_1-p_1}{2}\right)$$

$$+\left(\frac{-2}{\sin p_1}\right)\left(\sin\frac{q_1+s_1}{2}\sin\frac{q_1-s_1}{2}\right)+\left(\frac{-2}{\sin r_1}\right)\left(\sin\frac{p_1+s_1}{2}\sin\frac{p_1-s_1}{2}\right)$$

$$p,\ q,\ r,\ s\in\left(0,\ \frac{\pi}{2}\right)\times\left(0,\ \frac{\pi}{2}\right)\times\left(0,\ \frac{\pi}{2}\right)\times\left(0,\ \frac{\pi}{2}\right)<0$$

Clearly F is strictly invariant monotone with respect to η.

Let $p=\left(\frac{\pi}{6},\frac{\pi}{6},\frac{\pi}{6},\frac{\pi}{6}\right)$, $q=\left(\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{4}\right)$

Then $(q - p)^T(F(q) - F(p))$

$$=\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\left[\left(-1-\sin\frac{\pi}{4},\ -1-\sin\frac{\pi}{4},-1-\sin\frac{\pi}{4},-1-\sin\frac{\pi}{4}\right)\right.$$

$$\left.-\left(-1-\sin\frac{\pi}{6},-1-\sin\frac{\pi}{6},-1-\sin\frac{\pi}{6},\ -1-\sin\frac{\pi}{6}\right)\right]$$

$$=\frac{\pi}{12}\sqrt{4\left(\sin\frac{\pi}{6}-\sin\frac{\pi}{4}\right)^2}$$

$$=\frac{\pi}{6}\left(\frac{1}{2}-\frac{1}{\sqrt{2}}\right)$$

$$=\frac{\pi}{6}\left(\frac{1}{2}-\frac{\sqrt{2}}{2}\right)$$

$$=\frac{\pi}{6}\left(\frac{1-\sqrt{2}}{2}\right)<0$$

Thus F is not strictly monotone.

Remark

Every strictly invariant monotone map is an invariant monotone map with respect to the same η but the converse is not necessarily true.

Example

Define the map F and η as

$F(p) = (-\cos p_1, -\cos p_2, -\cos p_3, -\cos p_4)$

Where $p \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\eta(p, q, r, s) = \left[\frac{\sin p_1 - \sin q_1 - \sin r_1}{\cos s_1}, \frac{\sin p_2 - \sin q_2 - \sin r_2}{\cos s_2}, \frac{\sin p_3 - \sin q_3 - \sin r_3}{\cos s_3}, \frac{\sin p_4 - \sin q_4 - \sin r_4}{\cos s_4}\right]$$

$$p, q, r, s \in \left(-\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$$

$\eta(p, q, r, s)^T F(s) + \eta(s, p, q, r)^T F(r) + \eta(q, r, s, p)^T F(q) + \eta(r, s, p, q)^T F(p)$

$$= \frac{(\sin p_1 - \sin q_1 - \sin r_1)}{\cos s_1}(-\cos s_1) + \frac{(\sin p_2 - \sin q_2 - \sin r_2)}{\cos s_2}(-\cos s_2)$$

$$+ \frac{(\sin p_3 - \sin q_3 - \sin r_3)}{\cos s_3}(-\cos s_3) + \frac{(\sin p_4 - \sin q_4 - \sin r_4)}{\cos s_4}(-\cos s_4)$$

$$+ \frac{(\sin q_1 - \sin r_1 - \sin s_1)}{\cos p_1}(-\cos p_1) + \frac{(\sin q_2 - \sin r_2 - \sin s_2)}{\cos p_2}(-\cos p_2)$$

$$+ \frac{(\sin q_3 - \sin r_3 - \sin s_3)}{\cos p_3}(-\cos p_3) + \frac{(\sin q_4 - \sin r_4 - \sin s_4)}{\cos p_4}(-\cos p_4)$$

$$+ \frac{(\sin r_1 - \sin s_1 - \sin p_1)}{\cos q_1}(-\cos q_1) + \frac{(\sin r_2 - \sin s_2 - \sin p_2)}{\cos q_2}(-\cos q_2)$$

$$+ \frac{(\sin r_3 - \sin s_3 - \sin p_3)}{\cos q_3}(-\cos q_3) + \frac{(\sin r_4 - \sin s_4 - \sin p_4)}{\cos q_4}(-\cos q_4)$$

$$+ \frac{(\sin s_1 - \sin p_1 - \sin q_1)}{\cos r_1}(-\cos r_1) + \frac{(\sin s_2 - \sin p_2 - \sin q_2)}{\cos r_2}(-\cos r_2)$$

$$+ \frac{(\sin s_3 - \sin p_3 - \sin q_3)}{\cos r_3}(-\cos r_3) + \frac{(\sin s_4 - \sin p_4 - \sin q_4)}{\cos r_4}(-\cos r_4) = 0$$

Clearly F is invariant montone with respect to η.
Thus F is not strictly invariant monotone with respect to η.

Theorem

Let f and η satisfy Assumptions A and B respectively. Then f is a strictly preinvex function with respect to η on Γ if and only if ∇f is strictly invariant monotone with respect to η on Γ and f satisfies Assumption A.

Proof

Suppose f is strictly preinvex on Γ with respect to η, then

$f(p + \lambda\eta(p, q, r, s)) < \lambda f(p) + \lambda_1 f(q) + \lambda_2 f(r) + \lambda_3 (fs)$,

$f(q + \lambda\eta(s, p, q, r)) < \lambda f(s) + \lambda_1 f(p) + \lambda_2 f(q) + \lambda_3 f(r)$,

$f(r + \lambda\eta(q, r, s, p)) < \lambda f(q) + \lambda_1 f(r) + \lambda_2 f(s) + \lambda_3 f(p)$,

$f(s) + \lambda\eta(r, s, p, q)) < \lambda f(r) + \lambda_1 f(s) + \lambda_2 f(p) + \lambda_3 f(q)$.

First assumption A is just inequality with $\lambda = 1$, $\lambda_1 = \lambda_2 = \lambda_3 = 0$ in the above inequalities and lemma 2.1, it follows that f is strictly invariant monotone with respect to η on Γ.
Conversely, suppose that ∇f is strictly invariant monotone with respect η on Γ. Assume that f is not stricity preinvex with respect to η on Γ. Then there exist. p, q, r, s $\in \Gamma$ such that
$f(p + \eta(p, q, r, s)) \geq \lambda f(p) + \lambda_1 f(q) + \lambda_2 f(r) + \lambda_3 f(s)$,

where p, q, r, s $\in \Gamma$ such that $\lambda + \lambda_1 + \lambda_2 + \lambda_3 = 1$ and $\lambda \in [0, 1]$

Applying Mean Value Theorem,
$f(p + \lambda\eta) - f(p) = \lambda\eta f'(p + \theta\lambda\eta)$, when $0 < \theta < 1$

$$\Rightarrow \frac{f(p+\lambda\eta)-f(p)}{\lambda\eta} = f'(p+\theta\lambda\eta)$$

Now, f(p + λη(p, q, r, s) - f(p) ≥ (λ - 1) f(p) + λ_1f(q) + λ_2f(r) + λ_3f(s)

$$\Rightarrow \frac{f(p+\lambda\eta(p,\, q,\, r,\, s)-f(p)}{\lambda\eta} \geq \frac{(\lambda-1)f(p)+\lambda_1 f(q)+\lambda_2 f(r)+\lambda_3 f(s)}{\lambda\eta}$$

By using Equation ... (1), we get

$$f'(p+\theta\lambda\eta) \geq \frac{(\lambda-1)f(p)+\lambda_1 f(q)+\lambda_2 f(r)+\lambda_3 f(s)}{\lambda\eta}$$

$\Rightarrow \lambda\eta^T\nabla f(p) \geq (\lambda - 1)$ f(p) + λ_1f(q) + λ_2f(r) + λ_3f(s) ... (i)

Similarly,

$\lambda\eta^T\nabla f(q) \geq (\lambda_2 - 1)$ f(q) + λf(s) + λ_1f(p) + λ_3f(r) ... (ii)

$\lambda\eta^T\nabla f(r) \geq (\lambda_1 - 1)$f(r) + λf(q) + λ_2f(s) + λ_3f(p) ... (iii)

$\lambda\eta^T\nabla f(s) \geq (\lambda_1 - 1)$ f(s) + λf(r) + λ_2f(p) + λ_3f(q) ... (iv)

Adding Equation (i), (ii), (iii) and (iv), we have

$\eta(p, q, r, s)^T\nabla f(s) + \eta(s, p, q, r)^T\nabla f(r) + \eta(q, r, s, p)^T\nabla f(q) + \eta(r, s, p, q)^T\nabla f(p)$

$\geq ((\lambda - 1) + \lambda_1 + \lambda_2 + \lambda_3)\, f(p) + (\lambda_2 - 1 + \lambda_1 + \lambda + \lambda_3) f(q)$

$+ (\lambda_1 - 1 + \lambda_2 + \lambda_3 + \lambda) f(r) + (\lambda_1 - 1 + \lambda_2 + \lambda + \lambda_3)\, f(s) = 0$

which contradicts the strictly invariant monotonicity with respect to η on ∇f.

References

1. Avriel, M., Diewert, W.E., Schaible, S., and ZANG, I., Generalized Concavity, Plenum Press, New York, NY, 1988.
2. Hanson, M.A., On Sufficiency of the Kuhn Tucker conditions, *Journal of Mathematical Analysis and Applications*, Vol. 80, pp. 545-550, 1981.
3. Mohan, S.R., and Neogy, S.K., On Invex Sets and Preinvex Functions, *Journal of Mathematical Analysis and Applications*, Vol. 189, pp. 901-908, 1995.
4. Pradhan, S.K., Dalai, D.K., Dash, R.B., Generalized invexity and generalized invariant monotonicity, *Pacific-Asian Journal of Mathematics*, vol. 6, No.2, (July-December 2012)pp. 113-124.
5. Weir, T., and Mond, B., Preinvex Functions in Multiple-Objective Optimization, *Journal of Mathematical Analysis and Applications*, Vol. 136, pp. 29-38, 1988.
6. Weir, T., and Jeyakumar, V., A Class Of Nonconvex Functions and Mathematical Programming, *Bulletin of the Australian Mathematical Society*, Vol. 38, pp. 177-189, 1988.
7. Yang, X.M., Yang, X. Q., and TEO, K.L., Characterizations and Applications of Prequasiinvex Type Functions, *Journal of Optimization Theory and Applications*, Vol. 110, pp. 645-668, 2001.

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