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THREE SPECIAL SYSTEMS OF DOUBLE DIOPHANTINE EQUATIONS

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ABSTRACT

Three different pairs of Diophantine equations are considered for their corresponding non-zero distinct integer solutions. A few interesting properties among the solutions are presented.

INTRODUCTION

The pair of pell equations of the form $x^2 - ay^2 = 1$, $y^2 - bz^2 = 1$ where a and b are distinct non-square positive integers has been studied in (Anglin, 1996; Baker and Davenport, 1969; Walsh, 1997). In (Mihai, 2007), it has been proved that the system of equations $x^2 - (4m^2 - 1)y^2 = 1$, $y^2 - bz^2 = 1$ for positive integers m and b has atmost one positive integer solution. In (Fadwa S. Abu Muriefah and Amal al Rashed, 2006), it has been shown that the system of pell equations $y^2 - 5x^2 = 4$, $z^2 - 442x^2 = 441$ has no positive integer solutions. In this context, one may refer (Gopalan *et al*, 2014; Gopalan *et al*, 2016; Gopalan *et al*, 2016; Gopalan *et al*, 2016; Meena *et al*, 2016; Gopalan *et al*, 2016; Devibala *et al*, 2017). The above results motivated us to search for the integer solutions for some other choices of special double Diophantine equations. In particular, an attempt has been made to obtain non-zero distinct integer solutions to the three systems of double Diophantine equations namely, $y - x = p^{k+1}$, $y^2 + x^2 = (2s^2 - 2s + 1)p^{2k+4}$; $y - x = p^{2k}$, $y^3 - x^3 = (3s^2 - 3s + 1)p^{6k+2}$; $y - x = p^{2k+1}$, $y^3 - x^3 = (3s^2 - 3s + 1)p^{6k+5}$. A few interesting properties among the solutions are presented for each of the above systems.

Method of Analysis

System: 1

The system of double equations to be solved is

$$y - x = p^{k+1} \tag{1}$$

$$y^2 + x^2 = (2s^2 - 2s + 1)p^{2k+4}, \quad k \geq 1, s \geq 0 \tag{2}$$

Eliminating y between (1) and (2), the resulting equation is

$$2x^2 + 2p^{k+1}x + p^{2k+2} - (2s^2 - 2s + 1)p^{2k+4} = 0 \tag{3}$$

Treating (3) as a quadratic in x and solving for x, we have

$$x = \frac{1}{2}p^{k+1}\left[\sqrt{(4s^2 - 4s + 2)p^2 - 1} - 1\right] \tag{4}$$

Let $\alpha^2 = (4s^2 - 4s + 2)p^2 - 1$ (5)
whose smallest positive integer solution is

$$p(0) = 1, \quad \alpha(0) = 2s - 1$$

To find the other solutions of (5), consider the pellian

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$$\alpha^2 = \left(4s^2 - 4s + 2\right)p^2 + 1 \qquad (6)$$

whose smallest positive integer solution is

$$\tilde{p}(0) = 4s - 2, \quad \tilde{\alpha}(0) = 8s^2 - 8s + 3$$

The general solution $(\tilde{p}(n), \tilde{\alpha}(n))$ of (6) is given by

$$\tilde{\alpha}(n) = \frac{1}{2} f(n), \quad \tilde{p}(n) = \frac{1}{2\sqrt{4s^2 - 4s + 2}} g(n)$$

where

$$f(n) = \left(8s^2 - 8s + 3 + (4s-2)\sqrt{4s^2 - 4s + 2}\right)^{n+1} + \left(8s^2 - 8s + 3 - (4s-2)\sqrt{4s^2 - 4s + 2}\right)^{n+1}$$

$$g(n) = \left(8s^2 - 8s + 3 + (4s-2)\sqrt{4s^2 - 4s + 2}\right)^{n+1} - \left(8s^2 - 8s + 3 - (4s-2)\sqrt{4s^2 - 4s + 2}\right)^{n+1}$$

Applying the lemma of Brahmagupta between the solutions $(p(0), \alpha(0))$ and $(\tilde{p}(n), \tilde{\alpha}(n))$, we have

$$\alpha(n+1) = \frac{(2s-1)}{2} f(n) + \frac{1}{2}\sqrt{4s^2 - 4s + 2}\; g(n)$$

$$p(n+1) = \frac{1}{2} f(n) + \frac{(2s-1)}{2(4s^2 - 4s + 2)}\sqrt{4s^2 - 4s + 2}\; g(n)$$

In view of (4) and (1), the general values for x and y satisfying (1) and (2) are given by

$$\left.\begin{aligned} x(n+1) &= \frac{1}{2} p^{k+1}(n+1)\left(\alpha(n+1) - 1\right) \\ y(n+1) &= \frac{1}{2} p^{k+1}(n+1)\left(\alpha(n+1) + 1\right) \quad , n = -1, 0, 1, 2.... \end{aligned}\right\} \qquad (7)$$

To analyse the nature of solutions, one has to go for particular values for k and s. A few illustrations are given below.

Illustration

Let $k = 1, s = 2$ (8)

The corresponding system of equation under consideration is

$$y - x = p^2, \quad y^2 + x^2 = 5p^6 \qquad (9)$$

Using (8) in (7), the corresponding solutions to (9) are given by

$$x(n+1) = \frac{1}{2} p^2(n+1)\left(\alpha(n+1) - 1\right)$$

$$y(n+1) = \frac{1}{2} p^2(n+1)\left(\alpha(n+1) + 1\right)$$

where $p(n+1) = \frac{1}{2} f(n) + \frac{3\sqrt{10}}{20} g(n)$

$$\alpha(n+1) = \frac{3}{2} f(n) + \frac{\sqrt{10}}{2} g(n)$$

in which $f(n) = \left(19 + 6\sqrt{10}\right)^{n+1} + \left(19 - 6\sqrt{10}\right)^{n+1}$

$$g(n) = \left(19 + 6\sqrt{10}\right)^{n+1} - \left(19 - 6\sqrt{10}\right)^{n+1}$$

A few numerical examples are presented in Table: 1 below

Table 1 Numerical Examples

n	$x(n+1)$	$y(n+1)$	$p(n+1)$
-1	1	2	1
0	$(37)^2(58)$	$(37)^2(59)$	37
1	$(1405)^2(2221)$	$(1405)^2(2222)$	1405
2	$(53353)^2(84358)$	$(53353)^2(84359)$	53353
3	$(2026009)^2(3203401)$	$(2026009)^2(3203402)$	2026009

From the above table, we have the following observations

- $x(2n)$ *and* $y(2n+1)$ are odd whereas $x(2n+1)$ *and* $y(2n)$ are even, $n \geq 0$
- $\frac{2x(n+1)}{p^2(n+1)}$ *and* $\frac{2y(n+1)}{p^2(n+1)}$ are relatively prime.
- The sum $x(n+1) + y(n+1)$ is expressed as the difference of two squares.

Illustration

Let $k = 1, s = 3$ (10)

The corresponding system of equation under consideration is

$$y - x = p^2, \quad y^2 + x^2 = 13p^6 \qquad (11)$$

Using (10) in (7), the corresponding solutions to (11) are given by

$$x(n+1) = \frac{1}{2} p^2(n+1)\left(\alpha(n+1) - 1\right)$$

$$y(n+1) = \frac{1}{2} p^2(n+1)\left(\alpha(n+1) + 1\right)$$

where $p(n+1) = \frac{1}{2} f(n) + \frac{5}{2\sqrt{26}} g(n)$

$$\alpha(n+1) = \frac{5}{2} f(n) + \frac{\sqrt{26}}{2} g(n)$$

in which $f(n) = \left(51 + 10\sqrt{26}\right)^{n+1} + \left(51 - 10\sqrt{26}\right)^{n+1}$

$$g(n) = \left(51 + 10\sqrt{26}\right)^{n+1} - \left(51 - 10\sqrt{26}\right)^{n+1}$$

A few numerical examples are presented in Table: 2 below

Table 2 Numerical Examples

n	$x(n+1)$	$y(n+1)$	$p(n+1)$
-1	2	3	1
0	$(101)^2(257)$	$(101)^2(258)$	101
1	$(10301)^2(26262)$	$(10301)^2(26263)$	10301
2	$(1050601)^2(2678517)$	$(1050601)^2(2678518)$	1050601
3	$(107151001)^2(273182522)$	$(107151001)^2(273182523)$	107151001
4	$(10928351501)^2(27861938777)$	$(10928351501)^2(27861938778)$	10928351501

From the above table, we have the following observations

- The sum $x(n+1) + y(n+1)$ is expressed as the difference of two squares.

- $x(2n+1)\, and\ y(2n)$ are odd whereas $x(2n)\, and\ y(2n+1)$ are even, $n \geq 0$
- $\left(y^2(n+1)+x^2(n+1)\right)^2\left[13\left(y(n+1)-x(n+1)\right)+1\right]^2\left[26\left(y(n+1)-x(n+1)\right)-1\right] = 676\left(y^3(n+1)+x^3(n+1)\right)^2$

System 2

The system of double equations to be solved is

$$y - x = p^{2k} \tag{12}$$

$$y^3 - x^3 = \left(3s^2 - 3s + 1\right)p^{6k+2}, \quad k \geq 1, s \geq 0 \tag{13}$$

Eliminating y between (12) and (13), the resulting equation is

$$3p^{2k}x^2 + 3p^{4k}x + p^{6k} - \left(3s^2 - 3s + 1\right)p^{6k+2} = 0 \tag{14}$$

Treating (14) as a quadratic in x and solving for x, we have

$$x = \frac{1}{6}p^{2k}\left[\sqrt{\left(36s^2 - 36s + 12\right)p^2 - 3} - 3\right] \tag{15}$$

Let $\alpha^2 = \left(36s^2 - 36s + 12\right)p^2 - 3$ (16)

whose smallest positive integer solution is

$$p(0) = 1, \quad \alpha(0) = 6s - 3$$

To find the other solutions of (16), consider the pellian

$$\alpha^2 = \left(36s^2 - 36s + 12\right)p^2 + 1 \tag{17}$$

whose smallest positive integer solution is

$$\widetilde{p}(0) = 4s - 2, \quad \widetilde{\alpha}(0) = 24s^2 - 24s + 7$$

The general solution $\left(\widetilde{p}(n), \widetilde{\alpha}(n)\right)$ of (17) is given by

$$\widetilde{\alpha}(n) = \frac{1}{2}f(n), \quad \widetilde{p}(n) = \frac{1}{2\sqrt{36s^2 - 36s + 12}}g(n)$$

where

$$f(n) = \left(24s^2 - 24s + 7 + (4s-2)\sqrt{36s^2 - 36s + 12}\right)^{n+1} + \left(24s^2 - 24s + 7 - (4s-2)\sqrt{36s^2 - 36s + 12}\right)^{n+1}$$

$$g(n) = \left(24s^2 - 24s + 7 + (4s-2)\sqrt{36s^2 - 36s + 12}\right)^{n+1} - \left(24s^2 - 24s + 7 - (4s-2)\sqrt{36s^2 - 36s + 12}\right)^{n+1}$$

Applying the lemma of Brahmagupta between the solutions $\left(p(0), \alpha(0)\right)$ and $\left(\widetilde{p}(n), \widetilde{\alpha}(n)\right)$, we have

$$\alpha(n+1) = \frac{(6s-3)}{2}f(n) + \frac{1}{2}\sqrt{36s^2 - 36s + 12}\, g(n)$$

$$p(n+1) = \frac{1}{2}f(n) + \frac{(6s-3)}{2\sqrt{36s^2 - 36s + 12}}g(n)$$

In view of (15) and (12), the general values for x and y satisfying (12) and (13) are given by

$$\left.\begin{aligned} x(n+1) &= \frac{1}{6}p^{2k}(n+1)\left(\alpha(n+1) - 3\right) \\ y(n+1) &= \frac{1}{6}p^{2k}(n+1)\left(\alpha(n+1) + 3\right) \quad , n = -1, 0, 1, 2.... \end{aligned}\right\} \tag{18}$$

To analyse the nature of solutions, one has to go for particular values for k and s. A few illustrations are given below.

Illustration

Let $k = 1, s = 1$ (19)

The corresponding system of equation under consideration is

$$y - x = p^2, \quad y^3 - x^3 = p^8 \tag{20}$$

Using (19) in (18), the corresponding solutions to (20) are given by

$$x(n+1) = \frac{1}{6}p^2(n+1)\left(\alpha(n+1) - 3\right)$$

$$y(n+1) = \frac{1}{6}p^2(n+1)\left(\alpha(n+1) + 3\right)$$

where $p(n+1) = \frac{1}{2}f(n) + \frac{3}{2\sqrt{12}}g(n)$

$$\alpha(n+1) = \frac{3}{2}f(n) + \frac{\sqrt{12}}{2}g(n)$$

in which $f(n) = \left(7 + 2\sqrt{12}\right)^{n+1} + \left(7 - 2\sqrt{12}\right)^{n+1}$

$$g(n) = \left(7 + 2\sqrt{12}\right)^{n+1} - \left(7 - 2\sqrt{12}\right)^{n+1}$$

A few numerical examples are presented in Table: 3 below

Table 3 Numerical Examples

n	$x(n+1)$	$y(n+1)$	$p(n+1)$
-1	0	1	1
0	$(13)^2(7)$	$(13)^2(8)$	13
1	$(181)^2(104)$	$(181)^2(105)$	181
2	$(2521)^2(1455)$	$(2521)^2(1456)$	2521
3	$(35113)^2(20272)$	$(35113)^2(20273)$	35113
4	$(489061)^2(282359)$	$(489061)^2(282360)$	489061

From the above table, we have the following observations

- $x(2n+1)\, and\ y(2n)$ are odd whereas $x(2n)\, and\ y(2n+1)$ are even, $n \geq 0$
- The sum $x(n+1) + y(n+1)$ is expressed as the difference of two squares.
- $x(4n-3) \equiv 0 \pmod 7$, $n \geq 1$

Illustration

Let $k = 1, s = 2$ (21)

The corresponding system of equation under consideration is

$$y - x = p^2, \quad y^3 - x^3 = 7p^8 \tag{22}$$

Using (21) in (18), the corresponding solutions to (22) are given by

$$x(n+1)=\frac{1}{6}p^2(n+1)\left(\alpha(n+1)-3\right)$$

$$y(n+1)=\frac{1}{6}p^2(n+1)\left(\alpha(n+1)+3\right)$$

where $p(n+1)=\frac{1}{2}f(n)+\frac{9}{2\sqrt{84}}g(n)$

$$\alpha(n+1)=\frac{9}{2}f(n)+\frac{\sqrt{84}}{2}g(n)$$

in which $f(n)=\left(55+6\sqrt{84}\right)^{n+1}+\left(55-6\sqrt{84}\right)^{n+1}$

$$g(n)=\left(55+6\sqrt{84}\right)^{n+1}-\left(55-6\sqrt{84}\right)^{n+1}$$

A few numerical examples are presented in Table: 4 below

Table 4 Numerical Examples

n	$x(n+1)$	$y(n+1)$	$p(n+1)$
-1	1	2	1
0	$(109)^2(166)$	$(109)^2(167)$	109
1	$(11989)^2(18313)$	$(11989)^2(18314)$	11989
2	$(1318681)^2(2014318)$	$(1318681)^2(2014319)$	1318681
3	$(145042921)^2(221556721)$	$(145042921)^2(221556722)$	145042921

From the above table, we have the following observations

- $x(2n)$ *and* $y(2n+1)$ are odd whereas $x(2n+1)$ *and* $y(2n)$ are even, $n\geq 0$
- The sum $x(n+1)+y(n+1)$ is expressed as the difference of two squares.

System 3

The system of double equations to be solved is

$$y-x=p^{2k+1} \tag{23}$$

$$y^3-x^3=\left(3s^2-3s+1\right)p^{6k+5}, \quad k\geq 1, s\geq 0 \tag{24}$$

Eliminating y between (23) and (24), the resulting equation is

$$3p^{2k+1}x^2+3p^{4k+2}x+p^{6k+3}-\left(3s^2-3s+1\right)p^{6k+5}=0 \tag{25}$$

Treating (25) as a quadratic in x and solving for x, we have

$$x=\frac{1}{6}p^{2k+1}\left[\sqrt{\left(36s^2-36s+12\right)p^2-3}-3\right] \tag{26}$$

Following the procedure similar to system-2, the corresponding values for x and y satisfying (23) and (24) are given by

$$\left.\begin{aligned} x(n+1)&=\frac{1}{6}p^{2k+1}(n+1)\left(\alpha(n+1)-3\right)\\ y(n+1)&=\frac{1}{6}p^{2k+1}(n+1)\left(\alpha(n+1)+3\right) \quad, n=-1,0,1,2,\ldots\end{aligned}\right\} \tag{27}$$

To analyse the nature of solutions, one has to go for particular values for k and s. A few illustrations are given below.

Illustration

Let $k=1$, $s=1$ (28)

The corresponding system of equation under consideration is

$$y-x=p^3\,,\quad y^3-x^3=p^{11} \tag{29}$$

Using (28) in (27), the corresponding solutions to (29) are given by

$$x(n+1)=\frac{1}{6}p^3(n+1)\left(\alpha(n+1)-3\right)$$

$$y(n+1)=\frac{1}{6}p^3(n+1)\left(\alpha(n+1)+3\right)$$

where $p(n+1)=\frac{1}{2}f(n)+\frac{3}{2\sqrt{12}}g(n)$

$$\alpha(n+1)=\frac{3}{2}f(n)+\frac{\sqrt{12}}{2}g(n)$$

in which $f(n)=\left(7+2\sqrt{12}\right)^{n+1}+\left(7-2\sqrt{12}\right)^{n+1}$

$$g(n)=\left(7+2\sqrt{12}\right)^{n+1}-\left(7-2\sqrt{12}\right)^{n+1}$$

A few numerical examples are presented in Table: 5 below

Table 5 Numerical Examples

n	$x(n+1)$	$y(n+1)$	$p(n+1)$
-1	0	1	1
0	$(13)^3(7)$	$(13)^3(8)$	13
1	$(181)^3(104)$	$(181)^3(105)$	181
2	$(2521)^3(1455)$	$(2521)^3(1456)$	2521
3	$(35113)^3(20272)$	$(35113)^3(20273)$	35113
4	$(489061)^3(282359)$	$(489061)^3(282360)$	489061

From the above table, we have the following observations

- $x(2n+1)$ *and* $y(2n)$ are odd whereas $x(2n)$ *and* $y(2n+1)$ are even, $n\geq 0$
- The sum $x(n+1)+y(n+1)$ is expressed as the difference of two squares.
- $x(4n-3)\equiv 0(\mathrm{mod}\ 7)$, $n\geq 1$

Illustration

Let $k=1$, $s=2$ (30)

The corresponding system of equation under consideration is

$$y-x=p^3\,,\quad y^3-x^3=7p^{11} \tag{31}$$

Using (30) in (27), the corresponding solutions to (31) are given by

$$x(n+1)=\frac{1}{6}p^3(n+1)\left(\alpha(n+1)-3\right)$$

$$y(n+1)=\frac{1}{6}p^3(n+1)\left(\alpha(n+1)+3\right)$$

where $p(n+1)=\frac{1}{2}f(n)+\frac{9}{2\sqrt{84}}g(n)$

$$\alpha(n+1)=\frac{9}{2}f(n)+\frac{\sqrt{84}}{2}g(n)$$

in which $f(n)=\left(55+6\sqrt{84}\right)^{n+1}+\left(55-6\sqrt{84}\right)^{n+1}$

$g(n)=\left(55+6\sqrt{84}\right)^{n+1}-\left(55-6\sqrt{84}\right)^{n+1}$

A few numerical examples are presented in Table: 6 below

Table 6 Numerical Examples

n	$x(n+1)$	$y(n+1)$	$p(n+1)$
-1	1	2	1
0	$(109)^3(166)$	$(109)^3(167)$	109
1	$(11989)^3(18313)$	$(11989)^3(18314)$	11989
2	$(1318681)^3(2014318)$	$(1318681)^3(2014319)$	1318681
3	$(145042921)^3(221556721)$	$(145042921)^3(221556722)$	145042921

From the above table, we have the following observations

- $x(2n)$ *and* $y(2n+1)$ are odd whereas $x(2n+1)$ *and* $y(2n)$ are even, $n \geq 1$
- The sum $x(n+1)+y(n+1)$ is expressed as the difference of two squares.

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