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Research Article

FIXED POINT THEOREM IN CONE METRIC SPACE AND APPLICATION TO ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, we present a fixed point theorem for a generalised contraction in cone metric space. We also present an application to first-order ordinary differential equations with periodic boundary value problem and obtain the existence and uniqueness of a lower solution.

INTRODUCTION

Huang and Zhang [7] have introduced the concept of cone metric space by replacing the set of real numbers by an ordered Banach space and proved many fixed point theorems of contractive type mappings in cone metric space. Subsequently, many authors in [1], [15], [3],[9],[5],[6] have generalised the results of Huang and Zhang and have studied fixed point theorems for normal and non-normal cones. Fixed point theory has now evolved rapidly in cone metric space equipped with partial ordering. In [4], some results regarding partially ordered sets in cone metric spaces have been proved. In [11], [2] some results on the existence of fixed points for non-increasing function in cone metric space as well as an application to ordinary differential equations were found. In [12], some results on existence of a unique fixed point theorem for non-decreasing mappings are applied to obtain a unique solution for differential equation with boundary value problem. In this paper, we prove fixed point theorem for non-decreasing function satisfying generalised condition in cone metric space and also an application to ordinary differential equation was given.

Preliminaries

Definition

Let E be a real Banach space. A subset P of E is called a cone if and only if

1. P is closed, nonempty and P$\neq \{0\}$
2. a,b $\in$ R,a,b$\geq 0, x,y \in$P implies ax +by $\in$P
3. p$\cap(-$p)=$\{0\}$

Given a cone P $\subseteq$ E, we define a partial ordering $\leq$ with respect to P by $x \leq y$ implies $y - x \in$P. A cone P is called normal if there is a number k> 0 such that for all x,y $\in$ E,
0$\leq x \leq$y implies $\|x\| \leq k\|y\|$.

The least positive number satisfying the above inequality is called normal constant of P, while $x \ll y$ stands for$y - x \in$ int P(interior of P).

Definition

Let X be a nonempty set.Suppose that the mapping d:$X \times X \rightarrow$E satisfying

1. 0$\leq$d(x, y) for all $x, y \in X$ and d$(x, y) = 0$ if and only if $x = y$
2. d(x, y) =d(y, x) for all $x, y \in X$
3. d$(x, y) \leq$d(x, z) + d$(z, y)$for all $x, y, z \in X$

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Then d is called a cone metric in X and (X,d) is called a cone metric space.The concept of a cone metric space is more general than that of a metric space.

Definition

Let (X,d) be a cone metric space. We say that $\{x_n\}$ is

1. A Cauchy Sequence if for every c in E with c$\gg 0$, there is N such that for all n,m$> N$, d$(x_n, x_m)$$\ll$c
2. A Convergent Sequence if for every c in E with c$\gg 0$,there is N such that for all n$> N$, d(x_n, x) $\ll$c for some fixed x in X.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.It is known that $\{x_n\}$ convergent to x in X if and only if d$(x_n, x) \to 0\ as\ n \to \infty$.The limit of a convergent sequence is unique provided P is a normal cone with normal constant k.

In this paper, we consider a special case of the following periodic boundary value problem

u′(t) = $f(t, u(t))$ $\quad\quad if\ t \in I = [0, T].$ (1)

u(0)=u(T).

where T>0 and f : I× $R \to R$ is a continuous function.

Definition

A lower solution for (1) is a function $\beta \in C(I, R)$ such that $\beta\,'$(t) $\leq f(t, \beta(t))\ for\ t \in I, \beta(0) \leq \beta(T)$.
The following lemmas are used in main results.

Lemma 1.

Let P be a cone and $\{a_n\}$ be a sequence in X. If c$\in$ $int\ P\ and\ \theta \leq a_n \to\ \theta\ (as\ n \to \infty)$, then there exists N such that for all n$> N, we\ have\ a_n \ll c$.

Lemma 2.

Let $x, y, z \in X, if\ x \ll y\ and\ y\ \ll z, then\ x\ \ll z.$

Lemma 3.

Let P be a cone and $\theta \leq u \ll$ c for each c $\in int\ P$, then u= θ.

Main results

Theorem

Let (X,d) be a complete cone metric space. Let T: X $\to X$ be a increasing function satisfying the contractive condition

d(Tx, Ty) $\leq$a(d(x, y)) + $b[d(x, Tx) + d(y, Ty)]$ + $c[d(x, Ty) + d(y, Tx)]$

for $x, y \in X$ and the constants a,b,c $\in [0,1)$ $and a + b + c\ <$ $1.\ Then\ T\ has\ a\ unique\ fixed\ point\ in\ X.$

Proof

Let $x_0 \in X$. Consider $\{x_n\}$ where $x_n = Tx_{n-1}$; n≥ 1, $x_{n+1} = Tx_n$. From contraction,
d$(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1})$
$\leq$a((d(x_n, x_{n-1})) + $b[d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})]$ + $c[d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)]$
$\leq$a(d(x_n, x_{n-1})) + $b[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]$ + $c[d(x_n, x_n) + d(x_{n-1}, x_{n+1})]$
$\leq$a(d(x_n, x_{n-1})) + $b[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]$ + $c[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$
$\therefore$ d$(x_{n+1}, x_n)(1 - b - c) \leq d(x_n, x_{n-1})$(a+b+c)
$\Rightarrow$ d$(x_{n+1}, x_n)\ \leq d(x_n, x_{n-1})\,\frac{(a+b+c)}{(1-b-c)}$

Substituting $\ \ = \frac{(a+b+c)}{(1-b-c)}$ and as $0 \leq\ \ \leq 1$.
$\therefore$ d$(x_{n+1}, x_n)\ \leq\ \ d(x_n, x_{n-1}) \leq\ \ \leq\ \ ^{n}d(x_1, x_0)$
For any m≥ 1, $p \geq 1$, it follows that

d$(x_{m+p}, x_m)\ \leq d(x_{m+p}, x_{m+p-1}) +\ d(x_{m+p-1}, x_m)$
$\leq d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_{m+p-2})$+d$(x_{m+p-2}, x_m)$
$\leq$
$d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_{m+p-2})$+d$(x_{m+p-2}, x_{m+p-3})$ +….+d(x_{m+2}, x_{m+1})+d(x_{m+1}, x_m)
$\leq\ \ ^{m+p-1}$d(x_1, x_0)+ $^{m+p-2}$d(x_1, x_0)+ $^{m+p-3}$d(x_1, x_0)+ $\cdots$ $^{m+1}$d(x_1, x_0)+ ^{m} d(x_1, x_0)
$\leq$ ($^{m+p-1}$ + $^{m+p-2}$+ $^{m+p-3}$ + $\cdots$ $^{m+1}$+ ^{m})d(x_1, x_0)
$\leq \frac{^{m+1}}{(1-\ \ )}d(x_1, x_0)$ + ^{m}d(x_1, x_0)

For $\theta \ll$ c, $\frac{^{m+1}}{(1-\ \ )}$d(x_1, x_0) + ^{m}d$(x_1, x_0) \to\ \theta, as\ m \to \infty$
From Lemma 2.5, we find $m_0 \in N$, such that
$\frac{^{m+1}}{(1-\ \ )}$d(x_1, x_0) + ^{m}d$(x_1, x_0) \ll$ c
For each m$> m_0$. Hence,
d$(x_{m+p}, x_m) \leq \frac{^{m+1}}{(1-\ \ )}d(x_1, x_0)$ + ^{m}d$(x_1, x_0) \ll$ c

for all m$> m_0$ and any p. So by lemma $\{x_n\}$ is a Cauchy sequence in (X,d). Since (X,d) is a complete cone metric space, there exist $x^* \in X$, such that $x_n \to x^*$.

Let k$\gg\ \theta$ be arbitrary. Since $x_n \to x^*$ there exists N such that
d$(x_n, x^*) \ll \frac{k}{(\ \ +1)}\ \ for\ all\ n > N$
d$(x_n, x^*) \ll \frac{k}{\left(\frac{(a+b+c)}{(1-b-c)}+1\right)} = \frac{k(1-b-c)}{a-1}$ for $all\ n > N$
d$(x_n, x^*) \ll k$

Next we claim that x^* is a fixed point of T.
d(T$x^*, x^*)\ \leq d(Tx^*, Tx_n) + d(Tx_n, x^*)$
$\quad\quad = d(Tx^*, Tx_n) + d(x_{n+1}, x^*)$
d(T$x^*, x^*) \leq$a(d(x^*, x_n)) + $b[d(x^*, Tx^*) + d(x_n, Tx_n)]$ + $c[d(x^*, Tx_n) + d(x_n, Tx^*)]$+d$(x_{n+1}, x^*)$
$\leq$a(d(x^*, x_n)) + $b[d(x^*, Tx^*) + d(x_n, x_{n+1})]$ + $c[d(x^*, x_{n+1}) + d(x_n, Tx^*)]$+d$(x_{n+1}, x^*)$
$\leq$a(d(x^*, x_n)) + $b[d(x^*, Tx^*) + d(x_n, x^*) + d(x^*, x_{n+1})]$ + $c[d(x^*, x_{n+1}) + d(x_n, x^*) + d(x^*, Tx^*)]$+d$(x_{n+1}, x^*)$.
$\Rightarrow$d(T$x^*, x^*)\ (1 - b - c)\ \leq$d$(x^*, x_n)(a + b + c)$ + $d(x^*, x_{n+1})(1 + b + c)$
$\Rightarrow$d(T$x^*, x^*)\ \leq \frac{d(x^*, x_n)(a+b+c) + d(x^*, x_{n+1})(1+b+c)}{1-b-c}$
$\ll k.$

From Lemma 3, d(T$x^*, x^*) = \theta$, which implies x^* is a fixed point of T. Finally we need to prove the uniqueness of fixed point.

If there is another fixed point y^*, then

$d(x^*, y^*) = d(Tx^*, Ty^*)$
$\leq a\,(d(x^*, y^*)) + b[d(x^*, Tx^*) + d(y^*, Ty^*)]$
$\quad + c[d(x^*, Ty^*) + d(y^*, Tx^*)]$
$\leq d(x^*, y^*)(a + 2c)$
$\leq d(x^*, y^*)(1 - a - 2c) < 1$
$\Rightarrow d(x^*, y^*) < 1$.
Hence from Lemma 3, $x^* = y^*$.
Therefore, the proof is completed.

Application to ordinary differential equations

We consider here a special case of the following periodic boundary value problem

$$u'(t) = f(t, u(t)) \qquad if\ t \in I = [0, T]. \quad (1)$$

u(0)=u(T).
where T>0 and $f : I \times R \to R$ is a continuous function and suppose that there exists β>0 such that for $x, y \in R\ with\ y \geq x$

$$-\beta\{a(x-y) + b[(x - Tx) + (y - Ty)] + c[(x - Ty) + (y - Tx)]\} \leq [f(t,y) + \beta y] - [f(t,x) + \beta x)] \leq 0$$

Then the existence of lower solution $\alpha \in C(I, R)$ such that
$\alpha'(t) \leq f(t, \alpha(t))$ for all t$\in I \alpha(0) \leq \alpha(T)$

for (1) provides the existence of the unique solution to the problem.

Proof

The problem can be written in integral equation as
$u(t) = \int_0^T G(t,s)[f(s, u(s)) + \beta u(s)]ds$

$$\text{where} \quad G(t,s) = \begin{cases} \frac{e^{\beta(T+s-t)}}{e^{(\beta T - 1)}}, 0 \leq s \leq t \leq T \\ \frac{e^{\beta(s-t)}}{e^{(\beta T-1)}}, 0 \leq t \leq s \leq T \end{cases}$$

We consider the complete cone metric space X=C(I,R) with the distance$X = C(I,R)\ with\ the\ distance,$
$d(x,y) = sup|x(t) - y(t)|, x, y \in C(I, R)$.

We define the following order relation in, M=C(I,R) by $x, y \in C(I, R),\ x \leq y\ if\ and\ only\ if\ x(t) \leq y(t)$, for all $t \in I$.

We define H:C(I,R)→C(I,R) by
$[Hu](t) = \int_0^T G(t,s)[f(s, u(s)) + \beta u(s)]ds$
If u$\in C(I,R)\ is\ a\ fixed\ point\ of\ H, then\ u \in C^1(I$,R) is a solution of (1) for u$\geq v$,

$[Hu](t) = \int_0^T G(t,s)[f(s, u(s)) + \beta u(s)]ds$

$$\geq \int_0^T G(t,s)[f(s, u(s)) + \beta v(s)]ds$$

$=[Hv](t)$
Therefore H is increasing. For u$\geq v, we\ get$
$d(Hu, Hv) = sup|(Hu)(t) - (Hv)(t)|$

$$\leq sup \int_0^T G(t,s)[\{f(s, u(s)) + \beta u(s)\} - \{f(s, v(s) + \beta v(s)\}]ds$$

$$\leq \sup_{t \in I} \int_0^T G(t,s)\,\beta[ad(u,v) + b(d(u,Tu) + d(v,Tv)) + c(d(u,Tv) + d(v,Tu))$$

$$\leq \beta[ad(u,v) + b\big(d(u,Tu) + d(v,Tv)\big) + c(d(u,Tv) + d(v,Tu)] \times \frac{e^{(\beta T-1)}}{\beta e^{(\beta T-1)}}$$

and a lower solution to the problem is such that $\beta(t) \leq [H(\beta)](t)$, t$\in I$.

From theorem, finally H has a unique fixed point.

References

1. M.Abbas, B.E.Rhoades, Fixed and periodic point results in cone metric spaces, *Appl. Math. Letters*,22, No.4 (2009), 511-515.
2. A.Amini-Harandi, H.Emami, A fixed point theorem for contraction type maps in partially ordered metric space and application to ordinary differential equations, *Nonlinear Anal.* 72 (2010) 2238-2242.
3. I.Beg, A.Azam, M.Arshad, Common fixed points for maps on topological vector spaces, *Inter.J. Math. and Math.Science* 2009 (2009), 1-8.
4. L.Ciric, M. Abbas, R.Saadati and N.Hussain, Common fixed points of almost generalised contractive mappings in ordered metric spaces, *Applied mathematics and computation*, Vol.217, no. 12, pp.5784-5789.
5. A.K. Dubey and A.Narayan, Cone metric spaces and fixed point theorem for pair of constructive maps, *Int. J.Appl. Math.Research,* No.10(2012), 839-845.
6. A.K.Dubey, R.Shukla, R.P.Dubey, An extension of the paper cone metric spaces and fixed point theorems of contractive mappings, *Int. J.Appl. Math.Research*, No.1(2013),84-90.
7. L.G.Huang, X. Zhang, Cone Metric spaces and fixed point theorems f contractive mappings, *J.Math. Anal.Appl*.332(2) (2007), 1468-1476.
8. Hussain, N.Shah, MH; KKM mappings in cone b-metric spaces, *Comput.Math.Appl*.62, 1677-1684 (2011).
9. D.Ilic, V. Rakocevic, Common fixed points for maps on cone metric space, *J.Math.Anal.Appl*,341(2008),876-882
10. Jankovic,S. Kadelburg, Z. Radenovic, On Cone Metric Spaces: a survey, Nonlinear Anal.4(7), 2591-2601(2011).
11. Juan J.Nieto, Rosana Rodriguez-Lopez, Existence and uniqueness of fixed point in partially ordered sets and application to ordinary differential equations, *Acta Mathematica sinica, English series* Dec.2007,Vol.23, No.12, pp.2205-2212.
12. Nieto, J.J., Rodrigue Z-Lopez.R., Contractive Mapping Theorems inpartially ordered sets and application to ODE, Comput.*Math.Appl*, 22, 223-239 (2005).
13. Rezapour, S.Haklbarany, R. Somenotes on the paper, Cone Metric Spaces and fixed point theorems of contractive mappings, *J.Math.Anal.Appl*.345, 719-724(2008).
14. Ren. A.C.M. Reurings, M.C.B. A fixed point theorem in partially ordered sets applications to matrix equations. *Proc.Amer.Math.Soc*.,132,1435-1443 (2004).
15. R. Sumitra,V.R. Uthariaraj and R.Hemavathy, Common fixed point theorem for T-Hardy-Rogera. Contraction mapping in a cone metric space, *International Math. Forum* 5(2010) 1495-1506.