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Research Article

ON THE OF NILPOTENT ELEMENTS OF $\mathbb{Z}_n$ AS A RING AND $\mathbb{Z}_d$ AS A MODULE OVER $\mathbb{Z}_n$

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ABSTRACT

In this paper, we investigate the possible relationship between the nilpotent elements of $\mathbb{Z}_n$ as a ring and $\mathbb{Z}_d$ as a module over $\mathbb{Z}_n$. We provide examples of the fact that a nilpotent element in $\mathbb{Z}_n$ may not be nilpotent in the $\mathbb{Z}_n$ −module $\mathbb{Z}_d$ even if $d = n$. We give a condition under which a nilpotent element in $\mathbb{Z}_n$ becomes a nilpotent element in the $\mathbb{Z}_n -$ module $\mathbb{Z}_d$. We also obtain a partial converse of the above fact for the special case $d = n$ and under a certain condition.

INTRODUCTION

Let R be a ring. A non-zero element $a \in R$ is said to be nilpotent if there is some natural number r satisfying $a^r = 0$ but $a^{r-1} \neq 0$. In that case, we also say that r is the index of nilpotency of $a \in R$. Thus by definition, $0 \in R$ is trivially nilpotent. Almost every standard text on algebra, specifically on ring theory contains the above mentioned notion. For instance, one can go through [1], [3], [4].

In the year 2013, Groenewald and Ssiverrii introduced the notion of nilpotent elements of a module in [6]. They used the *negation* of the fact that an element $a \in R$ is not nilpotent if $a^2 = 0$ implies $a = 0$. Thus, an element $a \in R$ is nilpotent provided $a^2 = 0$ but $a \neq 0$. In the same spirit, a non-zero element m of a left R-module M is said to be nilpotent if $a^k m = 0$ but $am \neq 0$ for some $a \in R$ and $k \in \mathbb{N}$. By convention, we always consider $0 \in M$ to be a trivial nilpotent element. Any other nilpotent element of M is called a non-trivial nilpotent element. In the dissertation of the latter, the fact that a nilpotent element of a ring may not be the nilpotent element of the corresponding module has been exhibited. It is thus quite natural to investigate the nilpotent elements of $\mathbb{Z}_n$ as a ring and as a module over itself and of the $\mathbb{Z}_n -$ module $\mathbb{Z}_d$. We note that $\mathbb{Z}_d$ is a $\mathbb{Z}_n -$ module if $d|n$.

Throughout this work, we shall be concerned with elementary number theoretic results specially that of theory of congruence. For a familiarity with such results, one is referred to [2] and [5].

MAIN RESULTS

Throughout the section, $\mathbb{Z}_n$ will be considered as a ring and $\mathbb{Z}_d$ will be considered as a $\mathbb{Z}_n$ module.

We begin by pointing out that a nilpotent element of a ring may not be a nilpotent element of the ring when considered as a module over itself.

Example: The element $2 \in \mathbb{Z}_4$ is nilpotent when considered as a ring, but $2 \in \mathbb{Z}_4$ is not nilpotent when considered as a module over itself.

Theorem: *If $a \in \mathbb{Z}_n$ be a nilpotent element then a is a nilpotent element of the $\mathbb{Z}_n$ −module $\mathbb{Z}_d$ provided $a \in \mathbb{Z}_d$ and $a^2 \not\equiv 0(mod\ d)$.*

Proof: Let $a \in \mathbb{Z}_n$ be a nilpotent element. Then we have,

$a^k \equiv 0(modn)$

$\Rightarrow a^k \equiv 0(modd)\ \ as\ d|n$

This gives $a^{k-1}.a \equiv 0(modd)$. Also, by assumption, $a^2 = a.a \neq 0$. This shows that a is a nilpotent element of $\mathbb{Z}_d$.

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Example: $2 \in \mathbb{Z}_8$ is nilpotent as a ring element, but in the $\mathbb{Z}_8$-module $\mathbb{Z}_4$, 2 is not nilpotent as a module element. We observe that $2.2 \equiv 0(mod\ 4)$.

Example: $2 \in \mathbb{Z}_8$ is nilpotent as a ring element and $2 \in \mathbb{Z}_8$ is nilpotent in $\mathbb{Z}_8$ as a module over itself although $2^2 \not\equiv 0(mod\ 8)$.

We now give a couple of examples to illustrate the fact that the nilpotent element of a module may not be a nilpotent element when considered as a ring.

Example: The element $3 \in \mathbb{Z}_{12}$ is nilpotent when $\mathbb{Z}_{12}$ is considered as a module over itself. But $3 \in \mathbb{Z}_{12}$ is not nilpotent as a ring element.

Example: The element $3 \in \mathbb{Z}_8$ is nilpotent when $\mathbb{Z}_8$ is considered as a module over itself. But $3 \in \mathbb{Z}_8$ is not nilpotent when $\mathbb{Z}_8$ is considered as a ring.

We now provide a sufficient condition that establishes a connection between a nilpotent element of a module and a nilpotent element of the corresponding ring. The following theorem may also be viewed as a partial converse of Theorem 1.

Theorem: *Let $m \in \mathbb{Z}_n$ be a nilpotent element when $\mathbb{Z}_n$ is considered as a module over itself. The corresponding $a \in \mathbb{Z}_n$ that makes m nilpotent is itself nilpotent in $\mathbb{Z}_n$ when considered as a ring provided $(m, n) = 1$.*

Proof: Let $m \in \mathbb{Z}_n$ be nilpotent. Then, there exists $a \in \mathbb{Z}_n$ such that

$a^k m \equiv 0(modn)$ but $am \not\equiv 0\ (mod\ n)$

Also, $a^k m \equiv 0(modn)$ and $(m, n) = 1$ together implies that $a^k \equiv 0(modn)$. To show that $a \in \mathbb{Z}_n$ is nilpotent, all we need to show is that $a^{k-1} \not\equiv 0(modn)$. Let this be not the case. Then $a^{k-1} \equiv 0(modn)$ implies $a.a^{k-2} \equiv 0(modn)$ or $a^{k-2} \equiv 0(modn)$ as $a \not\equiv 0(modn)$. Repeating the above process, we may have $a^{k-3} \equiv 0(modn)$. Continuing in this way, we arrive at $a \equiv 0(modn)$. Thus, we have $am \equiv 0\ (mod\ n)$ which is a contradiction to the fact that $m \in \mathbb{Z}_n$ is nilpotent. Thus, $a^{k-1} \not\equiv 0(modn)$ and the theorem follows.

Example: $3 \in \mathbb{Z}_8$ is nilpotent as there exists $2 \in \mathbb{Z}_8$ satisfying $2^3.3 \equiv 0(mod8)$ but $2.3 \not\equiv 0(mod\ 8)$. We observe that $(3, 8) = 1$. Thus, by Theorem 2, $2 \in \mathbb{Z}_8$ is nilpotent when $\mathbb{Z}_8$ is considered as a ring.

Example: $4 \in \mathbb{Z}_9$ is nilpotent as there exists $3 \in \mathbb{Z}_9$ satisfying $3^2.4 \equiv 0(mod9)$ but $3.4 \not\equiv 0(mod\ 9)$. We observe that $(4, 9) = 1$. Thus, by Theorem 2, $3 \in \mathbb{Z}_9$ is nilpotent when $\mathbb{Z}_9$ is considered as a ring.

Theorem 2 gives a sufficient condition for an element of the ring $\mathbb{Z}_n$ to be nilpotent in connection with the nilpotent element of the corresponding module. However, the condition is not necessary at all.

Example: $2 \in \mathbb{Z}_8$ is nilpotent as there exists $2 \in \mathbb{Z}_8$ satisfying $2^2.2 \equiv 0(mod8)$ but $2.2 \not\equiv 0(mod\ 8)$. We observe that $(2, 8) \neq 1$ yet, $2 \in \mathbb{Z}_8$ is nilpotent when $\mathbb{Z}_8$ is considered as a ring.

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