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Research Article

STABILITY ANALYSIS OF FIRST ORDER DELAY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS USING INVERSE LAPLACE TRANSFORM

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ABSTRACT

This paper concerns the stability analysis of first order delay differential equations with constant coefficients. As stability is a very important problem in the theory and application of ordinary as well as delay differential equations and moreover stability analysis of delay differential equations have been investigated extensively, although not completely developed for more complicated cases yet, here we first introduce the concept of stability region and stability boundaries of first order delay differential equation with constant coefficients. Finally we approximate the characteristics equation of first order linear delay differential equation with determinant of square matrices as constant coefficients, using inverse Laplace transform as well as Gamma function to determine the stability of the delay differential equations.

INTRODUCTION

Delay differential equations (DDEs) are popular tools used by scientists in modeling real life systems. Delay Differential Equations (DDEs) are a large and important class of dynamical systems. DDEs are arise in either natural or technological problems. In these systems, a controller monitor the system and makes adjustment to the system based on its observation. Since these adjustments can never be made instantaneously, a delay arises between the observation and the control action.

It is well-known that stability is a very important problem in the theory and application of differential equations. If the solutions of a differential equation describing a dynamical system or of any differential equation under consideration are known in closed form, one can determine the stability properties of the system or the solution of the differential equation under consideration appealing directly the definitions of stability. Moreover finding of solutions become more difficult for delay differential equations rather than the differential equations without delay. The stability analysis of DDEs have been investigated extensively, although not completely developed for more complicated cases yet. Now there has been a recent surge of interest in numerical and analytical characterization of stability properties of linear Delay Differential Equations (DDEs). Stability analysis of DDEs is particularly relevant in control theory. The stability, as the basic requirement of control systems, may be destroyed due to the presence of time delay. The important objective of stability analysis is to find the maximal delay region such that time delay system remains stable for the time varying delay with this region. The determination of such region requires suitable stability criteria. Here the stability analysis of systems with time varying delays has been becoming a hot topic in the past few decades.

For the class of linear systems with constant coefficients and constant delay, the asymptotic analysis can be carried out by analyzing the characteristic equation. In case of scalar DDE, $y'(t) = ay(t) + by(t-\tau)$ a complete and satisfactory description of the stability region is obtained for a fixed delay, both for real and complex coefficients a and b. To date, the stability analysis of numerical methods for DDEs is almost

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completely restricted to equations with constant delay and to methods advancing with constant step size.

Some classical stability results for ordinary differential equation (ODE) methods

The simplest concept of stability for ODE methods is that, consider the linear autonomous test equation:

$$\left.\begin{aligned} y'(t) &= ay(t), \quad t \ge t_0 \\ y(t_0) &= y_0 \end{aligned}\right\} \qquad \ldots (1)$$

where $a \in C$.

The solution of (1), is given by

$$y(t) = e^{a(t-t_0)} y_0$$
$$\Rightarrow |y(t)| = \left|e^{a(t-t_0)}\right| \; |y_0|$$

$\Rightarrow |y(t)| = e^{R(a)(t-t_0)}|y_0|$, where R($a$) is the real part of $a \in C$

So, whenever $y_0 \ne 0$, the condition

$$R(a) \le a_0$$

is equivalent to $|y(t)| \le e^{a_0(t-t_0)} \,|\, y_0 \,|, \;\; t \ge t_0$

In particular, the condition R(a)<0 is equivalent to the asymptotic stability property,

$$\lim_{t\to+\infty} y(t) = 0$$

whereas the slightly weaker condition $R(a) \le \quad {}_0$ is equivalent to the contractivity property

$$|y(t)| \le |\, y_0 \,|, \;\; t \ge t_0$$

Linear Scalar test equations

Consider the class of scalar linear DDEs with variable coefficients

$$\left.\begin{aligned} y'(t) &= a(t) \;\; y(t) + b(t) y(t - \tau(t)), \quad t \ge t_0 \\ y(t) &= \phi(t), \;\; t \le t_0 \end{aligned}\right\} \qquad \ldots (2)$$

and with constant coefficients

$$\left.\begin{aligned} y'(t) &= a\, y(t) + b\;\, y(t - \tau(t)), \quad t \ge t_0 \\ y(t) &= \phi(t), \;\; t \le t_0 \end{aligned}\right\} \qquad \ldots (3)$$

If delay τ is constant, then equation (2) is a linear autonomous equation.
For the constant delay equation (3),

$$\left.\begin{aligned} y'(t) &= a\, y(t) + b\;\, y(t - \tau), \quad t \ge t_0 \\ y(t) &= \phi(t), \;\; t \le t_0 \end{aligned}\right\} \qquad \ldots (4)$$

The stability analysis may be done directly by studying the roots of the characteristic equation:

Let $y(t) = ce^{st}$

So, the characteristic equation for equation (4) is

$$cse^{st} = ace^{st} + bce^{s(t-\tau)}$$
$$\Rightarrow \;\; ce^{st}\left(s - a - b\,e^{-s\tau}\right) = 0$$
$$\Rightarrow \;\; s - a - b\,e^{-s\tau} = 0 \qquad \ldots(5)$$

It is known that (see El'sgol't and Norkin [7]), the equation (5) has infinitely many roots of s_i, each of which has a certain multiplicity m_i. They lie in the complex half plane $R(a) < \alpha$ for some real α and their real parts accumulate at $-\infty$. Therefore, in any vertical strip of the complex plane there are only a finite number of roots.

A necessary and sufficient condition for the asymptotic stability of equation (4) is that all the roots s_i of characteristic equation (5) such that $R(s_i) < 0$.

Asymptotic stability region S_τ for the real coefficients

We have the linear DDEs with constant coefficients

$$\left.\begin{aligned} y'(t) &= a\;\, y(t) + b\;\, y(t - \tau), \;\; t \ge t_0 \\ y(t) &= \phi(t), \;\; t \le t_0 \end{aligned}\right\} \qquad \ldots(6)$$

Let a and b are real.
The characteristic equation of (6) is

$$s - a - b\;\, e^{-s\tau} = 0 \qquad \ldots(7)$$

Analysis of the roots of the equation (7) shows that, for a fixed value of the delay τ, the region of stability S_τ is larger than the cone $a + |\, b \,| < 0$ derived by the inequality $R(a) + |\, b \,| < 0$ and the region of asymptotic stability S_τ is given by

$$S_\tau = \left\{ (a,b) :\; a < -b \, and \; \sqrt{b^2 - a^2} < \frac{1}{a}\cos^{-1}\left(\frac{-a}{b}\right) \right\}$$

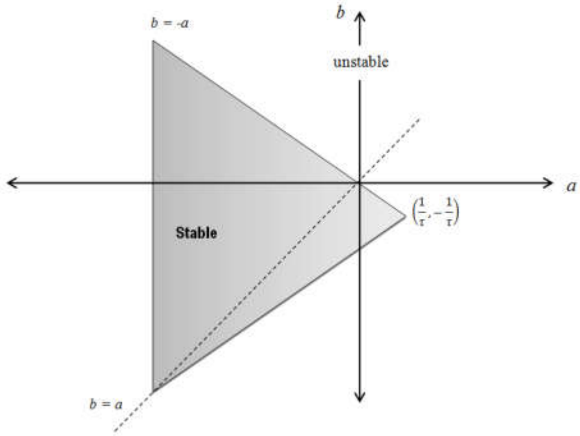

Fig 1 Asymptotic stability region S_τ for the equation (6) with constant delay τ in the real (a,b) plane.

The asymptotic stability condition

$$R(a)+|b|<0 \quad \ldots(8)$$

is not necessary for a fixed constant delay τ .When we assume the delay τ go to $+\infty$, we reveal that the region of asymptotic stability tends to the region described by

$$a<b<-a \quad \ldots(9)$$

Therefore, the condition(9) is slightly weaker than the condition(8)which is necessary for the asymptotic stability of equation (6) for all constant delay.

If we summarize the above stability conditions, it reflects in the following Table:

Table 1 Asymptotic stability scheme for $y'(t)=ay(t)+by(t-\tau)$, with $a,b\in R$.

$\lim\limits_{t\to\infty} y(t)=0$ for all constant delay τ
$\Leftrightarrow\ a\le b\le -a$
$\Leftrightarrow\ (a,b)\in S_\tau$
$\Leftrightarrow\ \lim\limits_{t\to+\infty} y(t)=0$, for fixed constant delay τ

First order delay differential equation with constant coefficients and stability boundaries

Consider
$$\left.\begin{aligned} y'(t)&=(\det A)y(t)+(\det B)y(t-\tau),\ t\ge 0\\ y(t)&=\phi(t),\ -\tau\le t\le 0\end{aligned}\right\} \quad \ldots(10)$$

where A and B are d×d matrices, $\tau>0$ and

$$\phi\in C^0\left([-\tau,0],R^d\right)$$

Also (det A) and (det B) are the determinants of square matrices A and B.

Let $\det A=a,\ \det B=b,\ \tau=1$

then equation (10) reduces to

$$\left.\begin{aligned} y'(t)&=a\ y(t)+b\ y(t-1), t\ge 0\\ y(t)&=\phi(t), -1\le t\le 0\end{aligned}\right\} \quad \ldots(11)$$

which is a first order linear DDE with constant coefficients.

Put $y(t)=c\ e^{st}$

$\Rightarrow y'(t)=cse^{st}$ and $y(t-1)=c\ e^{s(t-1)}$

then equation (11) becomes

$$cse^{st}=ace^{st}+bce^{s(t-1)}$$

$$\Rightarrow \quad s=a+be^{-s},\ \ e^{st}\ne 0$$

$$\Rightarrow \quad s-a-be^{-s}=0, \quad \ldots(12)$$

which is the transcendental characteristic equation of (10)

Put $s=iy,\ \ y\ge 0$ in equation (12), we get

$$iy-a-be^{-iy}=0$$

$$\Rightarrow \quad iy-a-b(\cos y-i\sin y)=0$$

($\because\ e^{iy}=\cos y+i\sin y$, which is the Euler's identity)

$$\Rightarrow \quad iy-a-b\cos y+ib\sin y)=0$$

$$\Rightarrow \left(-a-b\cos y\right)+i(y+b\sin y)=0$$

Equating real and imaginary parts, we get

$$-a-b\cos y=0 \quad \ldots(13)$$

$$y+b\sin y=0 \quad \ldots(14)$$

From equations (13) and (14), we have

$a=-b\cos y$, $\quad b\sin y=-y$

$$\Rightarrow \quad a=-b\cos y,\quad b=-y\cos ec\ y$$

$$\Rightarrow \quad a=y\cot y,\quad b=-y\ \cos ec\ \ y$$

when $a=0$, from equation (13), $b\cos y=0$

$\Rightarrow \cos y=0$, provided $b\ne 0$

$$\Rightarrow y=\frac{\pi}{2}$$

when $y=\frac{\pi}{2}$, from equation (14), $\frac{\pi}{2}=-b\sin\frac{\pi}{2}$

$$\Rightarrow b=-\frac{\pi}{2}$$

So, when $a=0, b=\frac{\pi}{2}$,

when $b=0,\ a=0$ and

when $y=0$, from equation (13), $b=-a$.

Thus, we see that a zero root $y=0$ occurs on the line $b=-a$

For non-zero y, the stability curve (or family of curves) is of the form

$$S_n=\{(a,b)=(y\cot y,-y\cos ec\ y), n\pi<y<(n+1)\pi\}, n\ge 0 \quad \ldots(15)$$

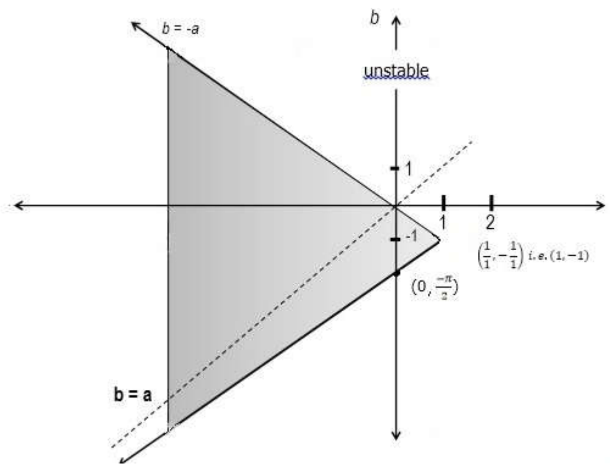

Fig 2 Analytical stability chart of equation (11)

Here the Fig : 2 shows a portion (a,b) parameter space with the analytically stability boundary, i.e, in which $y(t)=0$. Solution of equation (11) is marginally stable.

The upper and lower stability curves intersect at (1,-1). The lower stability curve S_0 meets the b-axis at $\left(0, \frac{-\pi}{2}\right)$ and then approaches to $(-\infty,-\infty)$ along the line $b = a$.
The shaded region represents asymptotically stable solution of equation (11).

Polynomial approximations of the characteristic equation of first order linear DDEs with stability

Consider the linear first order DDE with constant coefficients

$$\left.\begin{aligned} &y'(t) = (\det A)y(t) + (\det B)y(t-\tau),\ t \geq 0 \\ &y(t) = \phi(t),\ -\tau \leq t \leq 0 \end{aligned}\right\} \quad \text{... (16)}$$

where A and B are $d \times d$ matrices, τ > 0 and $\phi \in c^0\left([-\tau,0], R^d\right)$.

Also (det A) and (det B) are the determinants of square matrices A and B.

$Let\ \det A = a, \det B = b\ and\ \tau = 1$

Then equation (16) becomes

$$\left.\begin{aligned} &y'(t) = ay(t) + by(t-1),\ t \geq 0 \\ &y(t) = \phi(t), -1 \leq t \leq 0 \end{aligned}\right\} \quad \text{...(17)}$$

From equation (17),

$$y'(t) = ay(t) + by(t-1)$$

$$\Rightarrow \quad y'(t) - ay(t) = by(t-1) \quad \text{...(18)}$$

Taking the Laplace transform of both sides of the equation (18),

$$L\{y'(t)\} - a\ L\{y(t)\} = b\,L\{y(t-1)\},\ t \in [0,1)$$

(using the Linearity property of Laplace transform)

$$\Rightarrow \quad [s\,L\{y(t)\} - y(0)] - a\ L\{y(t)\} = b\ L(\phi)$$

($\because$ when t=0, $L\{f'(t)\} = s\,L\{f(t)\} - f(0)$)

$$\Rightarrow \quad s\ L(y) - y_0 - a\ L(y) = b\ L(\phi)$$

($\because$ y_0 is the initial condition at $t = 0$)

$$\Rightarrow \quad (s-a)L(y) - y_0 = b\,L(\phi) \quad \text{...(19)}$$

The Laplace transform on function f is defined as

$$L(f) = \int_0^1 e^{-st} f(t)\ dt \quad \text{... (20)}$$

and this is equivalent to taking the Laplace transform of the function extended by zero on $(1,\infty)$.

From equation (19),

$$(s-a)\ L(y) = b\ L(\phi) + y_0$$

$$\Rightarrow L(y) = \frac{b\ L(\phi) + y_0}{s-a}$$

For the first interval [0,1], the Laplace transform $y(t)$ can be expressed as

$$L(y) = \frac{b\,L(\phi) + y_0}{s-a} \quad \text{...(21)}$$

At the end of the interval $[0,1], L(y)$ can be calculated by evaluating the inverse Laplace transform of $L(y)$ at $t = 1$ denoted by L_1^{-1}.

i.e, $y_1 = L^{-1}(L(y))(1) = L_1^{-1}(L(y))$ ...(22)

Let $\quad Y_0(s) = L(\phi),\ Y_1(s) = L(y)$ ...(23)

Equations (21) and (22) can be written as

$$Y_1(s) = \frac{bY_0(s) + y_0}{s-a} \quad \text{...(24)}$$

$$y_1 = L_1^{-1}(y_1(s)) \quad \text{...(25)}$$

Similarly,

$$Y_2(s) = \frac{bY_1(s) + y_1}{s-a} \quad \text{... (26)}$$

$$y_2 = L_1^{-1}(Y_2(s)) \quad \text{... (27)}$$

In general,

$$Y_n(s) = \frac{bY_{n-1}(s) + y_{n-1}}{s-a} \quad \text{...(28)}$$

$$y_n = L_1^{-1}(Y_n(s)) \quad \text{... (29)}$$

From the above sequence of approximations,

$$Y_{n-1}(s) = \frac{bY_{n-2}(s) + y_{n-2}}{s-a}$$

$$y_{n-1} = L_1^{-1}(Y_{n-1}(s))$$

Putting the value of $Y_{n-1}(s)$ in equation (28) gives

$$Y_n(s) = \frac{b\left(\frac{bY_{n-2}(s) + y_{n-2}}{s-a}\right) + y_{n-1}}{s-a}$$

$$\Rightarrow \quad Y_n(s) = \frac{y_{n-1}}{s-a} + \frac{by_{n-2}}{(s-a)^2} + \frac{b^2 Y_{n-2}(s)}{(s-a)^2} \quad \text{... (30)}$$

Again, using

$$Y_{n-2}(s) = bY_{n-3}(s) + y_{n-3}$$

$$y_{n-2} = L_1^{-1}(Y_{n-2}(s))$$

in equation (30) , gives

$$Y_n(s) = \frac{y_{n-1}}{s-a} + \frac{by_{n-2}}{(s-a)^2} + \frac{b^2\left(\frac{(bY_{n-3}(s) + y_{n-3})}{s-a}\right)}{(s-a)^2}$$

$$\Rightarrow \quad Y_n(s) = \frac{y_{n-1}}{s-a} + \frac{b\ y_{n-2}}{(s-a)^2} + \frac{b^2 y_{n-3}}{(s-a)^3} + \frac{b^3 Y_{n-3}(s)}{(s-a)^3}$$

The repeated application of the procedure terminates at Y_0 and we get,

$$Y_n(s) = \frac{y_{n-1}}{s-a} + \frac{b\, y_{n-2}}{(s-a)^2} + \frac{b^2 y_{n-3}}{(s-a)^3} + \ldots + \frac{b^{n-1} y_0}{(s-a)^n} + \frac{b^n Y_0(s)}{(s-a)^n} \quad \ldots(31)$$

(Using previous equations and lastly using equation (21))

$$\Rightarrow Y_n(s) = \left[\frac{b^{n-1} y_0}{(s-a)^n} + \frac{b^{n-2} y_1}{(s-a)^{n-1}} + \ldots + \frac{b^2 y_{n-3}}{(s-a)^3} + \frac{b y_{n-2}}{(s-a)^2} + \frac{y_{n-1}}{s-a} \right]$$

$$+ \frac{b^n Y_0(s)}{(s-a)^n}$$

$$\Rightarrow Y_n(s) = \left(\sum_{i=0}^{n-1} \frac{b^{n-1-i}}{(s-a)^{n-i}} y_i \right) + b^n \frac{Y_0(s)}{(s-a)^n}$$

Again, from equation (29),

$$y_n = L_1^{-1}(Y_n(s))$$

$$\Rightarrow y_n = L_1^{-1}\left(\frac{bY_{n-1}(s) + y_{n-1}}{s-a} \right), \quad \text{(using equation (28))}$$

$$\Rightarrow y_n = L_1^{-1}\left[\frac{b\left(\frac{bY_{n-2}(s) + y_{n-2}}{s-a} \right) + y_{n-1}}{s-a} \right]$$

$$\Rightarrow y_n = L_1^{-1}\left[\frac{y_{n-1}}{s-a} + \frac{by_{n-2}}{(s-a)^2} + \frac{b^2 Y_{n-2}(s)}{(s-a)^2} \right]$$

and repeating the process, we get y_n in terms of $y_0, y_1, \ldots, y_{n-1}$

i.e,

$$y_n = L_1^{-1}\left[\left(\frac{y_{n-1}}{s-a} + \frac{by_{n-2}}{(s-a)^2} + \frac{b^2 y_{n-3}}{(s-a)^3} + \ldots + \frac{b^{n-1} y_0}{(s-a)^n} \right) + \frac{b^n Y_{n-n}(s)}{(s-a)^n} \right]$$

$$\Rightarrow$$

$$y_n = L_1^{-1}\left[\left(\frac{b^{n-1} y_0}{(s-a)^n} + \frac{b^{n-2} y_1}{(s-a)^{n-1}} + \ldots + \frac{b^1 y_{n-2}}{(s-a)^2} + \frac{b^0 y_{n-1}}{(s-a)^1} \right) \right] + L_1^{-1}\left(\frac{b^n Y_0(s)}{(s-a)^n} \right)$$

$$\Rightarrow y_n = L_1^{-1}\left(\sum_{i=0}^{n-1} \frac{b^{n-1-i} y_i}{(s-a)^{n-i}} \right) + b^n L_1^{-1}\left(\frac{Y_0(s)}{(s-a)^n} \right) \quad \ldots(32)$$

As $Y_0(s) = L(\phi)$ is the initial function and stability should not depend on the initial function, so we neglect the inverse Laplace term $L_1^{-1}\left(\frac{Y_0(s)}{(s-a)^n} \right)$ in equation (32) we get,

$$y_n \cong \sum_{i=0}^{n-1} L_1^{-1}\left(\frac{1}{(s-a)^{n-i}} \right) b^{n-1-i} y_i \quad \ldots(33)$$

(Using the linearity property of inverse Laplace transform)

For positive integer *n*, the equation (33) depends on all previous conditions. We know

$$L\{f(t)\} = \bar{f}(s)$$

$$\Rightarrow L^{-1}\{\bar{f}(s)\} = f(t)$$

Also, $$L^{-1}\left\{ \frac{1}{(s-a)^n} \right\} = \frac{e^{at} t^{n-1}}{(n-1)!}, \; n = 1,2,3,\ldots$$

In the first interval [0,1],

$$L_1^{-1}\left\{ \frac{1}{(s-a)^n} \right\} = \frac{e^{a\times 1} \; (1)^{1-1}}{(n-1)!} = \frac{e^a}{(n-1)!} \quad \ldots(34)$$

Using equation (34) in equation (33) , we get

$$y_n = \sum_{i=0}^{n-1} e^a \frac{1}{(n-i-1)!} b^{n-1-i} y_i$$

$$\Rightarrow y_n = e^a \sum_{i=0}^{n-1} \frac{1}{(n-i-1)!} b^{n-1-i} y_i \quad \ldots(35)$$

To obtain characteristic equation of the Laplace transform of equation (35),

Put $x_i = \lambda^i, \; \lambda \neq 0$

So equation (35) becomes,

$$\lambda^n = e^a \sum_{i=0}^{n-1} \frac{1}{(n-1-i)!} b^{n-1-i} \lambda^i$$

$$\Rightarrow \lambda^n - e^a \sum_{i=0}^{n-1} \frac{1}{(n-1-i)!} b^{n-1-i} \; \lambda^i = 0 \quad \ldots(36)$$

Taking the index $n-1-i$ as j i.e, $j = n-1-i$ we get,

$$\lambda^n - e^a \sum_{j=0}^{n-1} \frac{1}{j!} \; b^j \; \lambda^{n-1-j} = 0$$

[when $i = 0, \; j = n-1$; $i = n-1, \; j = 0$]

$$\Rightarrow \lambda^n - e^a \sum_{j=0}^{n-1} \frac{1}{j!} \; b^j \; \lambda^{n-1} \; \lambda^{-j} = 0$$

$$\Rightarrow \lambda^n - e^a \; \lambda^{n-1} \sum_{j=0}^{n-1} \frac{1}{j!} \left(\frac{b}{\lambda} \right)^j = 0$$

$$\Rightarrow \lambda^n - e^a \; \lambda^{n-1} \sum_{j=0}^{n-1} \frac{\left(\frac{b}{\lambda} \right)^j}{j!} = 0 \quad \ldots(37)$$

As $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

So, we can write,

$$\sum_{j=0}^{\infty} \frac{\left(\frac{b}{\lambda} \right)^j}{j!} = e^{\frac{b}{\lambda}} \quad \ldots(38)$$

We have the Gamma function defined as

$$\Gamma(n,a) = \int_a^{\infty} t^{n-1} e^{-t} dt \text{ and } \Gamma(n) = (n-1)!$$

So, $$\Gamma(n, \frac{b}{\lambda}) = \int_{b/\lambda}^{\infty} t^{n-1} e^{-t} dt$$

$$=\left[-t^{n-1}e^{-t}\right]_{b/\lambda}^{\infty}+\int_{b/\lambda}^{\infty}(n-1)t^{n-2}e^{-t}\ dt$$

$$=\left[0+\left(\frac{b}{\lambda}\right)^{n-1}e^{-(b/\lambda)}\right]+(n-1)\left[\left[-t^{n-2}e^{-t}\right]_{b/\lambda}^{\infty}+\int_{b/\lambda}^{\infty}(n-2)t^{n-3}e^{-t}dt\right]$$

$$=\left(\frac{b}{\lambda}\right)^{n-1}e^{-(b/\lambda)}+(n-1)\left[0+\left(\frac{b}{\lambda}\right)^{n-2}e^{-(b/\lambda)}\right]+(n-1)n-2)\left[\left[-t^{n-3}e^{-t}\right]_{b/\lambda}^{\infty}+\int_{b/\lambda}^{\infty}(n-3)t^{n-3}e^{-t}dt\right]$$

$$=\left(\frac{b}{\lambda}\right)^{n-1}e^{-(b/\lambda)}+(n-1)\left(\frac{b}{\lambda}\right)^{n-2}e^{-(b/\lambda)}+(n-1)(n-2)\left(\frac{b}{\lambda}\right)^{n-3}e^{-(b/\lambda)}+(n-3)\int_{b/\lambda}^{\infty}t^{n-3}e^{-t}dt$$

Continuing the process (i.e, repeated application of integration byparts) we get,

$$\Gamma\left(n,\frac{b}{\lambda}\right)=\left(\frac{b}{\lambda}\right)^{n-1}e^{-(b/\lambda)}+(n-1)\left(\frac{b}{\lambda}\right)^{n-2}e^{-(b/\lambda)}+(n-1)(n-2)\left(\frac{b}{\lambda}\right)^{n-3}e^{-(b/\lambda)}+\ldots$$
$$+(n-1)(n-2)\ldots n-(n-1)\left(\frac{b}{\lambda}\right)^{n-n}e^{-(b/\lambda)}$$

$$\Rightarrow\Gamma\left(n,\frac{b}{\lambda}\right)=e^{-(b/\lambda)}\left[\left(\frac{b}{\lambda}\right)^{n-1}+(n-1)\left(\frac{b}{\lambda}\right)^{n-2}+(n-1)(n-2)\left(\frac{b}{\lambda}\right)^{n-3}+\ldots\right.$$
$$+(n-1)(n-2)\ldots.n-(n-2)\left(\frac{b}{\lambda}\right)^{1}$$
$$\left.+(n-1)(n-2)\ldots.n-(n-1)\left(\frac{b}{\lambda}\right)^{0}\right]$$

$$\Rightarrow\Gamma\left(n,\frac{b}{\lambda}\right)=e^{-(b/\lambda)}\left[\left(\frac{b}{\lambda}\right)^{n-1}+(n-1)\left(\frac{b}{\lambda}\right)^{n-2}+(n-1)(n-2)\left(\frac{b}{\lambda}\right)^{n-3}+\ldots\right.$$
$$+(n-1)(n-2)\ldots 2\left(\frac{b}{\lambda}\right)^{1}$$
$$\left.+(n-1)(n-2)\ldots.1\left(\frac{b}{\lambda}\right)^{0}\right]$$

$$\Rightarrow\Gamma\left(n,\frac{b}{\lambda}\right)=e^{-(b/\lambda)}\left[(n-1)!\left(\frac{b}{\lambda}\right)^{0}+\frac{(n-1)!}{1!}\left(\frac{b}{\lambda}\right)^{1}+\frac{(n-1)!}{2!}\left(\frac{b}{\lambda}\right)^{2}+\ldots\right.$$
$$\left.+\frac{(n-1)!}{(n-2)!}\left(\frac{b}{\lambda}\right)^{n-2}+\frac{(n-1)!}{(n-1)!}\left(\frac{b}{\lambda}\right)^{n-1}\right]$$

$$\Rightarrow\Gamma\left(n,\frac{b}{\lambda}\right)=e^{-(b/\lambda)}(n-1)!\left[\frac{\left(\frac{b}{\lambda}\right)^{0}}{0!}+\frac{\left(\frac{b}{\lambda}\right)^{1}}{1!}+\frac{\left(\frac{b}{\lambda}\right)^{2}}{2!}+\ldots.+\frac{\left(\frac{b}{\lambda}\right)^{n-2}}{(n-2)!}+\frac{\left(\frac{b}{\lambda}\right)^{n-1}}{(n-1)!}\right]$$

$$\Rightarrow\frac{\Gamma\left(n,\frac{b}{\lambda}\right)}{(n-1)!}=e^{-(b/\lambda)}\left[\frac{\left(\frac{b}{\lambda}\right)^{0}}{0!}+\frac{\left(\frac{b}{\lambda}\right)^{1}}{1!}+\frac{\left(\frac{b}{\lambda}\right)^{2}}{2!}+\ldots+\frac{\left(\frac{b}{\lambda}\right)^{n-2}}{(n-2)!}+\frac{\left(\frac{b}{\lambda}\right)^{n-1}}{(n-1)!}\right]$$

$$\Rightarrow\frac{\Gamma\left(n,\frac{b}{\lambda}\right)}{\Gamma(n)}=e^{-(b/\lambda)}\sum_{j=0}^{n-1}\frac{\left(\frac{b}{\lambda}\right)^{j}}{j!}$$

$$\Rightarrow\frac{\Gamma\left(n,\frac{b}{\lambda}\right)}{\Gamma(n)}=e^{-(b/\lambda)}\sum_{j=0}^{n-1}\frac{\left(\frac{b}{\lambda}\right)^{j}}{j!}$$

$$\Rightarrow e^{\frac{b}{\lambda}}\frac{\Gamma\left(n,\frac{b}{\lambda}\right)}{\Gamma(n)}=\sum_{j=0}^{n-1}\frac{\left(\frac{b}{\lambda}\right)^{j}}{j!}$$

$$\Rightarrow\sum_{j=0}^{n-1}\frac{\left(\frac{b}{\lambda}\right)^{j}}{j!}=e^{(b/\lambda)}\frac{\Gamma\left(n,\frac{b}{\lambda}\right)}{\Gamma(n)}\qquad\ldots(39)$$

The partial sum in equation (37) can be written as,

$$\lambda^{n}-e^{a}\lambda^{n-1}e^{(b/\lambda)}\frac{\Gamma\left(n,\frac{b}{\lambda}\right)}{\Gamma(n)}=0$$

$$\Rightarrow\lambda^{n}-\lambda^{n-1}e^{a}\,e^{(b/\lambda)}\ \frac{\Gamma\left(n,\frac{b}{\lambda}\right)}{\Gamma(n)}=0$$

$$\Rightarrow\lambda^{n}-\lambda^{n-1}e^{\left(a+\frac{b}{\lambda}\right)}\ \frac{\Gamma\left(n,\frac{b}{\lambda}\right)}{\Gamma(n)}=0$$

Thus we have ,

$$f_{n}(\lambda)=\lambda^{n}-\lambda^{n-1}e^{\left(a+\frac{b}{\lambda}\right)}\ \frac{\Gamma\left(n,\frac{b}{\lambda}\right)}{\Gamma(n)}=0\qquad\ldots(40)$$

which is an n[th] order polynomial approximation to determine the stability of equation (16)

Here, we replaced the original stability problem with that of a difference equation and

$f_{n}(\lambda)\to 0$ when $n\to\infty$ and $-1<\lambda<1$

So the condition for stability is $|\lambda|<1$

i.e, the equation is stable when all roots of characteristic equation are on the unit circle.

Convergence of nth order polynomial approximation to transcendental function

We have in equation (40),

$$f_{n}(\lambda)=\lambda^{n}-\lambda^{n-1}e^{\left(a+\frac{b}{\lambda}\right)}\frac{\Gamma\left(n,\frac{b}{\lambda}\right)}{\Gamma(n)}=0$$

When $n\to\infty$, the equation (39) becomes

$$e^{\frac{b}{\lambda}}\ \frac{\Gamma\left(n,\frac{b}{\lambda}\right)}{\Gamma(n)}=\sum_{j=0}^{\infty}\frac{\left(\frac{b}{\lambda}\right)^{j}}{j}$$

$$\Rightarrow e^{\frac{b}{\lambda}}\ \frac{\Gamma\left(n,\frac{b}{\lambda}\right)}{\Gamma(n)}=e^{\frac{b}{\lambda}}$$

$$\Rightarrow \frac{\Gamma\left(n,\frac{b}{\lambda}\right)}{\Gamma(n)}=1$$

So, $\lambda^{n}-\lambda^{n-1}e^{\left(a+\frac{b}{\lambda}\right)}\frac{\Gamma\left(n,\frac{b}{\lambda}\right)}{\Gamma(n)}=0$

$$\Rightarrow \frac{\lambda^{n}}{\lambda^{n-1}}-e^{\left(a+\frac{b}{\lambda}\right)}\ \frac{\Gamma\left(n,\frac{b}{\lambda}\right)}{\Gamma(n)}=0$$

$$\Rightarrow \lambda-e^{\left(a+\frac{b}{\lambda}\right)}\ \frac{\Gamma\left(n,\frac{b}{\lambda}\right)}{\Gamma(n)}=0$$

when $n\rightarrow\infty,\frac{\Gamma\left(n,\frac{b}{\lambda}\right)}{\Gamma(n)}=1$

$$\Rightarrow \lambda-e^{\left(a+\frac{b}{\lambda}\right)}=0 \qquad \ldots(41)$$

Also, the characteristic equation of the original continuous problem in eq (10) is

$$s-a-be^{-s}=0$$

$$\Rightarrow s=a+be^{-s} \qquad \ldots(42)$$

Let $\lambda=\dfrac{b}{s-a}$

$$\Rightarrow\ s\lambda-a\lambda=b$$

$$\Rightarrow\ s=a+\frac{b}{\lambda}$$

So equation (42) becomes

$$a+\frac{b}{\lambda}=a+b\ \ e^{-\left(a+\frac{b}{\lambda}\right)}$$

$$\Rightarrow\ \frac{b}{\lambda}=b\ \ e^{-\left(a+\frac{b}{\lambda}\right)}$$

$$\Rightarrow \frac{1}{\lambda}=e^{-\left(a+\frac{b}{\lambda}\right)}$$

$$\Rightarrow \lambda=e^{a+\frac{b}{\lambda}} \qquad \ldots(43)$$

which is equivalent to equation (41). Hence the characteristic equation of equation (10) is equivalent to the n[th] order polynomial approximation of the same equation (10) and the sequence of nth order polynomial approximation $\{f_n(\lambda)\},\ n\in N$ converges to transcendental function $e^{a+\frac{b}{\lambda}}$.

CONCLUSION

In this paper inverse Laplace transform is introduced for stability analysis of DDEs. The stability analysis is dependent on stability region S_τ and the convergence properties of the method are studied elegantly and it further can be implemented while discussing stability analysis of DDEs.

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