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Research Article

MATHEMATICAL MODELING AND ANALYSIS OF GROUND WATER INFILTRATION IN UNSATURATED POROUS MEDIA BY USING POWER SERIES METHOD (PSM)

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ABSTRACT

In this paper, we have discussed groundwater infiltration phenomenon in unsaturated porous media in horizontal direction. The governing equation has been obtained in the form of second order non linear diffusion equation by using Darcy's law. The solution is obtain in the form of infinite series by using Power Series Method (PSM). The numerical as well graphical presentation is given by MAT LAB coding.

INTRODUCTION

Groundwater is derived from rain and melting snow that percolate downward from the surface; it collects in the open pore spaces between soil particles or in cracks and fissures in bedrock. The process of percolation is called infiltration. Most drinking water supplies and often irrigation water for agricultural needs are drawn from underground sources. More than 90 percent of the liquid fresh water available on or near the earth's surface is groundwater.

When fluid in filtered in porous medium (unsaturated soil), its velocity decreases as soil becomes saturated such phenomenon is called infiltration. The infiltration model was first developed by (Boussinesq, 1903) and is related to the original motivation of (Darcy, 1856; Polubarinova Kochina, 1962; Scheidegger, 1968; Muskat, 1946; Bear, 1972). Different researchershave discussed this problem with a different point of views. (Verma, 1967) discussed infiltration of incompressible fluid for inclined plain in heterogeneous porous media. (Mehta, 1977) obtained the solution of singular perturbation technique of one dimensional flow in unsaturated porous media. (Parikh and Mehta, 2011) discussed the atmospheric pressure in dry region and velocity of in filtered water in ground water infiltration phenomenon. (Borana and Pradhan, 2013) obtained the numerical solution of boussinesq's equation arising in one-dimensional infiltration phenomenon by using finite difference method. (Patel and Mehta, 2014) obtained a solution of boussinesq's equation for infiltration phenomenon in unsaturated porous media by homotopy analysis method. (Pathak and Singh, 2015) obtained an analytic solution of mathematical model of boussinesq's equation in homogeneous porous media during infiltration of ground water flow. (Desai, 2017) obtained similarity solution of non-linear boussinesq's equation arising in infiltration of incompressible fluid flow.

In this paper, we have studied groundwater infiltration phenomenon in unsaturated porous media in horizontal direction. The governing equation is nonlinear second order diffusion equation which has been solved by using Power Series Method (PSM). Our purpose is to find height of the free surface of the water mound, the atmospheric pressure in dry region and velocity of in filtered ground water at different distance and different time.

Mathematical Formulation

The basic assumptions to develop the mathematical model of the groundwater infiltration phenomenon are: The stratum has

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height h_{max} and lies on top of a horizontal impervious bed, which welabel as $z=0$, ignore the transversal variable y and the water mass which infiltrates the soil occupies a region described as$\Omega=\{(x,z)\in R: z\le h(x,t)\}$.(Vazquez 2007)

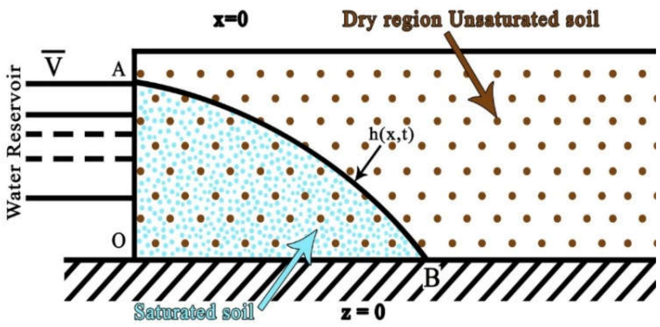

Figure 1 Schematic diagram of Groundwater Infiltration

In practical terms, we assume that there is no partial saturation. Clearly, $0\le h(x,t)\le h_{max}$ where $h_{max}$is the maximum height of free surface and $h(x,t)$ is an unknown free boundary function. For the sake of simplicity and for the practical computation after introducing a suitable assumption, the hypothesis of almost horizontal flow, i.e., we assumethat the flow has an almost horizontal speed. Here $u\approx(u,0)$, so that h has small gradients. It follows that the vertical component of the momentum equation is

$$\rho\left(\frac{\partial u_z}{\partial t}\right)+u\cdot\nabla u_z=-\frac{\partial p}{\partial z}-\rho g \tag{1}$$

Neglecting the inertial term (the left-hand side) of equation (1), and integrating with respect to z, we obtain first approximation $p+\rho gz=$ constant. Now calculate the constant on the free surface $z=h(x,t)$. If we impose continuity of the pressure across the interface, we have $p=0$ (assuming constant atmospheric pressure in the air that fills the pores of the dry region $z>h(x,t)$), we get

$$p=\rho g(h-z) \tag{2}$$

In other words, the pressure is determined by means of the hydrostatic approximation. Consider the mass conservation law for a section $S=(x,x+a)\times(0,C)$, we get

$$\phi\frac{\partial}{\partial t}\int_x^{x+a}\int_0^h dy\,dx=-\int_{\partial s}u\cdot n\;dl \tag{3}$$

where ϕ the porosity of the medium, i.e., the fraction of volume available for the flow circulation, and u is the velocity, which obeys Darcy's law in the form that includes gravity effects

$$u=-\frac{k}{\mu}\nabla(p+\rho gz) \tag{4}$$

On the right-hand lateral surface we have $u\cdot n\approx(u,0)\cdot(1,0)=u$i.e. $-\left(\frac{k}{\mu}\right)\frac{\partial p}{\partial x}$ while on the left-hand side we have$-u$. Using the formula for p and differentiating with respect to x, we get

$$a\phi\frac{\partial h}{\partial t}=\frac{\rho gk}{\mu}\frac{\partial}{\partial x}\int_0^h\frac{\partial}{\partial x}h\,dz \tag{5}$$

Thus, we obtained non-linear second order diffusion equation (Boussinesq's equation) as

$$\frac{\partial h}{\partial t}=\frac{\rho gk}{a\mu\phi}\frac{\partial^2}{\partial x^2}h^2 \tag{6}$$

with an initial and boundary condition as

$$h(x,0)=h(x),\qquad t=0,x>0 \tag{7}$$

$$h(0,t)=h_{max},\qquad x=0,t>0 \tag{8}$$

An equivalent form of equation (6) is

$$\frac{\partial h}{\partial t}=\beta\left[h\frac{\partial^2 h}{\partial x^2}+\left(\frac{\partial h}{\partial x}\right)^2\right], \tag{9}$$

where, $\beta=\frac{2\rho gk}{a\mu\phi}$.

Choosing new dimensionless variables as

$$X=\frac{x}{L},T=\frac{\beta t}{L^2} \tag{10}$$

Then equation (9) is in the following form

$$\frac{\partial h}{\partial T}=h\frac{\partial^2 h}{\partial X^2}+\left(\frac{\partial h}{\partial X}\right)^2, \tag{11}$$

with an initial and boundary condition as

$$h(X,0)=e^{-X},\qquad T=0,X>0 \tag{12}$$

$$h(0,T)=1,\qquad X=0,T>0 \tag{13}$$

Solution by Power Series Method (PSM)

Let us assume the solution of equation (11) as a power series in X and T as given below

$$h(X,T)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_{ij}\,X^iT^j \tag{14}$$

Differentiating equation (14) partially with respect to X and T, we get series expansion of $\frac{\partial h}{\partial X}$ and $\frac{\partial h}{\partial T}$are as follows:

$$\frac{\partial h}{\partial X}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(i+1)\,c_{(i+1)j}X^iT^j \tag{15}$$

$$\frac{\partial h}{\partial T}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(j+1)\,c_{i(j+1)}X^iT^j \tag{16}$$

Also the series expansion of $\left(\frac{\partial h}{\partial X}\right)^2,\frac{\partial^2 h}{\partial X^2}$ and $h\,\frac{\partial^2 h}{\partial X^2}$ are as follows:

$$\left(\frac{\partial h}{\partial X}\right)^2=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\sum_{s=0}^{j}\sum_{t=0}^{i}(t+1)(i-t+1)\,c_{(t+1)s}c_{(i-t+1)(j-s)}\right]X^iT^j \tag{17}$$

$$\frac{\partial^2 h}{\partial X^2}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(i+1)(i+2)\,c_{(i+2)j}X^iT^j \tag{18}$$

$$h\,\frac{\partial^2 h}{\partial X^2}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\sum_{s=0}^{j}\sum_{t=0}^{i}(i-t+1)(i-t+2)\,c_{ts}c_{(i-t+2)(j-s)}\right]X^iT^j \tag{19}$$

Substituting equations (16), (17) and (19) into (11), to obtain the recurrence relation:

$$c_{i(j+1)}=\frac{1}{(j+1)}\left[\sum_{s=0}^{j}\sum_{t=0}^{i}\Big((i-t+1)(i-t+2)c_{ts}c_{(i-t+2)(j-s)}+(t+1)(i-t+1)c_{(t+1)s}c_{(i-t+1)(j-s)}\Big)\right],\forall\, i,j\geq 0 \tag{20}$$

The initial condition (12) should be transformed as follows,

$$\sum_{i=0}^{\infty}c_{i0}\,X^i=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}X^i \tag{21}$$

which gives

$$c_{i0}=\frac{(-1)^i}{i!},\qquad \forall\, i\geq 0 \tag{22}$$

and the boundary condition (13) should be transformed as follows,

$$c_{0j}=\begin{cases}1, if\ j=0\\ 0, if\ j\geq 1\end{cases} \tag{23}$$

By applying the recurrence relation (20) for several values of i and j the approximate series solution obtained is as follows,

$$h(X,T)=1-X+\frac{1}{2}X^2-\frac{1}{6}X^3+\frac{1}{24}X^4-\frac{1}{120}X^5+\cdots-4XT+4X^2T-\frac{8}{3}X^3T+\frac{4}{3}X^4T-\cdots-26XT^2+40X^2T^2-\frac{121}{3}X^3T^2+\cdots-\frac{656}{3}XT^3+\cdots \tag{24}$$

Changing into original variables,

$$h(x,t)=1-\left(\frac{x}{L}\right)+\frac{1}{2}\left(\frac{x}{L}\right)^2-\frac{1}{6}\left(\frac{x}{L}\right)^3+\frac{1}{24}\left(\frac{x}{L}\right)^4-\frac{1}{120}\left(\frac{x}{L}\right)^5+\cdots-4\left(\frac{x}{L}\right)\left(\frac{\beta}{L^2}\right)t+4\left(\frac{x}{L}\right)^2\left(\frac{\beta}{L^2}\right)t-\frac{8}{3}\left(\frac{x}{L}\right)^3\left(\frac{\beta}{L^2}\right)t+\frac{4}{3}\left(\frac{x}{L}\right)^4\left(\frac{\beta}{L^2}\right)t-\cdots-26\left(\frac{x}{L}\right)\left(\frac{\beta}{L^2}\right)^2t^2+40\left(\frac{x}{L}\right)^2\left(\frac{\beta}{L^2}\right)^2t^2-\frac{121}{3}\left(\frac{x}{L}\right)^3\left(\frac{\beta}{L^2}\right)^2t^2+\cdots-\frac{656}{3}\left(\frac{x}{L}\right)\left(\frac{\beta}{L^2}\right)^3t^3+\cdots \tag{25}$$

By using equations (2) and (25), the atmospheric pressure in the air that fills the pores of dry region$z>h(x,t)$ is

$$p=\rho g\left[1-\left(\frac{x}{L}\right)+\frac{1}{2}\left(\frac{x}{L}\right)^2-\frac{1}{6}\left(\frac{x}{L}\right)^3+\frac{1}{24}\left(\frac{x}{L}\right)^4-\frac{1}{120}\left(\frac{x}{L}\right)^5+\cdots-4\left(\frac{x}{L}\right)\left(\frac{\beta}{L^2}\right)+4\left(\frac{x}{L}\right)^2\left(\frac{\beta}{L^2}\right)t-\frac{8}{3}\left(\frac{x}{L}\right)^3\left(\frac{\beta}{L^2}\right)t+\frac{4}{3}\left(\frac{x}{L}\right)^4\left(\frac{\beta}{L^2}\right)t-\cdots-26\left(\frac{x}{L}\right)\left(\frac{\beta}{L^2}\right)^2t^2+40\left(\frac{x}{L}\right)^2\left(\frac{\beta}{L^2}\right)^2t^2-\frac{121}{3}\left(\frac{x}{L}\right)^3\left(\frac{\beta}{L^2}\right)^2t^2+\cdots-\frac{656}{3}\left(\frac{x}{L}\right)\left(\frac{\beta}{L^2}\right)^3t^3+\cdots-z\right] \tag{26}$$

Also by using equations (2) and (4), the velocity of infiletered water is

$$u=-\frac{k\rho g}{\mu}\frac{\partial h}{\partial x} \tag{27}$$

Differentiating equation (25) partially with respect to x and then putting the value of $\frac{\partial h}{\partial x}$ in equation (27), we get

$$u=\frac{k\rho g}{\mu L}\left[1-\left(\frac{x}{L}\right)+\frac{1}{2}\left(\frac{x}{L}\right)^2-\frac{1}{6}\left(\frac{x}{L}\right)^3+\frac{1}{24}\left(\frac{x}{L}\right)^4-\cdots+4\left(\frac{\beta}{L^2}\right)t-8\left(\frac{x}{L}\right)\left(\frac{\beta}{L^2}\right)t+8\left(\frac{x}{L}\right)^2\left(\frac{\beta}{L^2}\right)t-\frac{16}{3}\left(\frac{x}{L}\right)^3\left(\frac{\beta}{L^2}\right)t+\cdots+26\left(\frac{\beta}{L^2}\right)^2t^2-80\left(\frac{x}{L}\right)\left(\frac{\beta}{L^2}\right)^2t^2+121\left(\frac{x}{L}\right)^2\left(\frac{\beta}{L^2}\right)^2t^2-\cdots+\frac{656}{3}\left(\frac{\beta}{L^2}\right)^2t^2-\cdots\right] \tag{28}$$

Numerical and Graphical Representation

Numerical and graphical presentations of an equations (25), (26) and (28) have been obtained by using MAT LAB. The following tables (1), (2) and (3) represents the numerical values for height, pressure and velocity for different distance x for fixed time $t=0.1, 0.2, 0.3, 0.4, 0.5$, respectively. Also figures (2), (3) and (4) shows the graph of Height h vs. Distance x, Pressure p vs. Distance x and Velocity u vs. Distance x for fixed time $t=0.1, 0.2, 0.3, 0.4, 0.5$, respectively.

Table 1 Numerical values for h corresponding to x and t, taking β=0.1,g=9.8,ρ=0.1are fixed

	h				
x	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$	$t = 0.5$
0.0	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	0.9010	0.8965	0.8914	0.8854	0.8784
0.2	0.8117	0.8036	0.7942	0.7833	0.7705
0.3	0.7312	0.7202	0.7072	0.6921	0.6743
0.4	0.6586	0.6451	0.6291	0.6103	0.5881
0.5	0.5931	0.5774	0.5588	0.5366	0.5102
0.6	0.5340	0.5165	0.4954	0.4700	0.4394
0.7	0.4807	0.4615	0.4381	0.4094	0.3745
0.8	0.4327	0.4121	0.3862	0.3541	0.3145
0.9	0.3894	0.3676	0.3394	0.3035	0.2588
1.0	0.3505	0.3277	0.2971	0.2572	0.2068

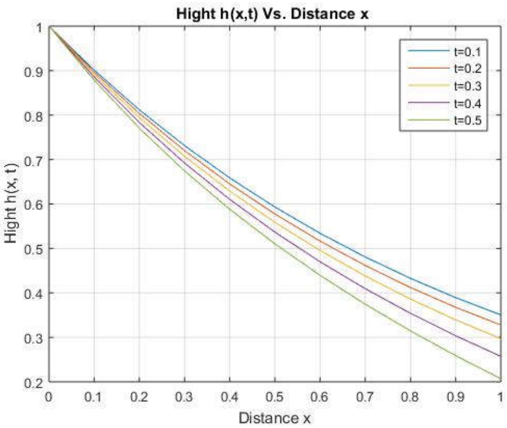

Figure 2 Hight h at different distance x with a fix time level $t = 0.1, 0.2, 0.3, 0.4, 0.5$

Table 2 Numerical values for p corresponding to x and t, taking $\beta = 0.1, g = 9.8, \rho = 0.1$& $z = 0.1$are fixed

	p				
x	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$	$t = 0.5$
0.0	0.8820	0.8820	0.8820	0.8820	0.8820
0.1	0.7849	0.7806	0.7755	0.7696	0.7628
0.2	0.6975	0.6896	0.6804	0.6696	0.6571
0.3	0.6186	0.6077	0.5951	0.5803	0.5628
0.4	0.5474	0.5342	0.5186	0.5001	0.4783
0.5	0.4832	0.4679	0.4496	0.4279	0.4020
0.6	0.4253	0.4081	0.3875	0.3626	0.3326
0.7	0.3731	0.3543	0.3313	0.3032	0.2690
0.8	0.3260	0.3058	0.2805	0.2490	0.2103
0.9	0.2836	0.2622	0.2346	0.1994	0.1557
1.0	0.2455	0.2232	0.1931	0.1541	0.1047

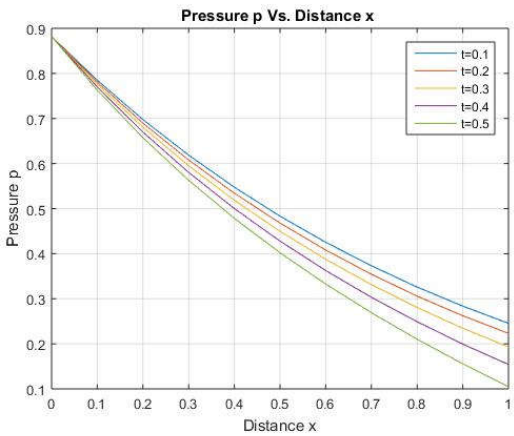

Figure 3 Pressure p at different distance x with a fix time level $t = 0.1, 0.2, 0.3, 0.4, 0.5$

Table 3 Numerical values for u corresponding to x and t, taking $\beta = 0.1, g = 9.8, \rho = 0.1$ are fixed

	$\boldsymbol{u}$				
x	$t = 0.1$	$t = 0.2$	$t = 0.3$	$t = 0.4$	$t = 0.5$
0.0	0.0102	0.0107	0.0113	0.0119	0.0127
0.1	0.0092	0.0096	0.0101	0.0106	0.0112
0.2	0.0083	0.0086	0.0090	0.0094	0.100
0.3	0.0075	0.0078	0.0081	0.0085	0.0089
0.4	0.0068	0.0070	0.0073	0.0076	0.0080
0.5	0.0061	0.0063	0.0065	0.0069	0.0073
0.6	0.0055	0.0057	0.0059	0.0062	0.0066
0.7	0.0050	0.0051	0.0053	0.0057	0.0061
0.8	0.0045	0.0046	0.0048	0.0052	0.0057
0.9	0.0040	0.0041	0.0044	0.0047	0.0053
1.0	0.0036	0.0037	0.0039	0.0043	0.0049

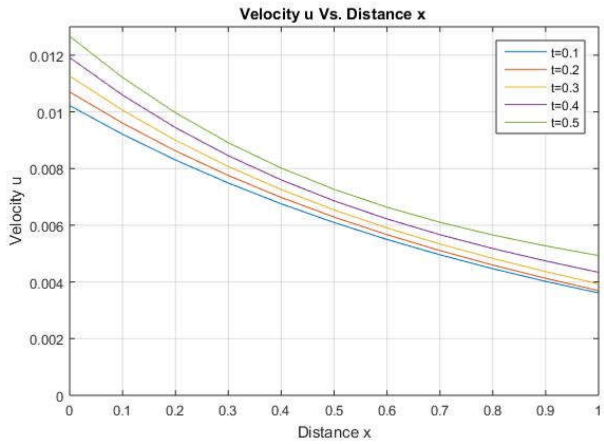

Figure 4 Velocity u at different distance x with a fix time level $t = 0.1, 0.2, 0.3, 0.4, 0.5$

CONCLUSION

In this paper, we have studied groundwater infiltration phenomenon in unsaturated homogeneous porous media in horizontal direction. Mathematical formulation has been done by using Darcy's law, the mass conservation law and the momentum equations. Equation (9) represents the governing equation of groundwater infiltration phenomenon which is known as Boussinesq's equation. The solution of governing equation (9) is obtained in equation (25) as an infinite series which represents height of water mound. It can be observed form table (1) and figure (2) that the height of water mound is decreasing with respect to distance as well as time. The solution of the atmospheric pressure in the air is obtained in (26). It can be observed form table (2) and figure (3) that the atmospheric pressure in the air is decreasing with respect to distance as well as time. The velocity of infiletered water is obtained in (27). It can be observed form table (3) and figure (4) that the velocity of infiletered water is decreasing with respect to distance and increasing with respect to time.

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