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Research Article

THE DEPENDENCE OF BLOW-UP TIME WITH RESPECT TO PARAMETERS FOR SMALL REACTION DIFFUSION EQUATION

Yekre B*., Yoro.Gozo and Halima Nachid B

Université Nangui Abrogoua, UFR-SFA, Département de Mathématiques et Informatiques, (Côte d'Ivoire)

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ABSTRACT

In this paper we consider the following initial-boundary problem.

$$\begin{cases} u_t(x,t)-\gamma u_{xx}(x,t)=f\left(u(x,t)\right) & x \text{ in}\left(0,1\right),\ t\text{ in}\left(0,T\right), \\ u(0,t)=0,\ u(1,t)=0, & t\text{ in}\left[0,T\right], \\ u(x,0)=u_0(x), & x \text{ in } \left[0,1\right]. \end{cases}$$

Where $f(s)$ is a positive, increasing, convex function for nonnegative value of S, $f(0)\succ 0$, $\int_0^{+\infty}\frac{ds}{f(s)}<+\infty$, and γ is a positive diffusion parameter. We find some conditions under which the solution of semi-discrete form of the above problem blows up in a finite time and estimate its semi-discrete blow up time. We also prove the convergence of the semidiscrete form blow-up time to the real one when the mesh size tends to zero. Finally, we give some numerical results to illustrate our analysis.

INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider the following initial-boundary value problem for a nonlinear equation with a nonlinear boundary condition of the form

$$u_t(x,t)-\gamma u_{xx}(x,t)=f\left(u(x,t)\right) \quad x \text{ in}\left(0,1\right),\ t\text{ in}\left(0,T\right), \tag{1}$$

$$u(0,t)=0,\ u(1,t)=0, \quad t\text{ in}\left[0,T\right], \tag{2}$$

$$u(x,0)=u_0(x), \quad x \text{ in}\left[0,1\right]. \tag{3}$$

which models the temperature distribution of a large number of physical phenomena from physics, chemistry and biology. The initial data $u_0(x)$ is a continious and increasing function in $\left[0,1\right]$, f(s) is a positive, increasing, convex function for nonnegative values of s, $f(0)\succ 0$, $\int_0^{+\infty}\frac{ds}{f(s)}<+\infty$, and γ is a positive diffusion parameter.

Here (0, *T*) us Here (0, *T*) is the maximal time interval of existence of the solution u. The time *T* may be finite or infinite. When *T* is in_nite, we say that the solution *u* exists globally. When T is finite, then the solution u develops a singularity in a finite time, namely, $\lim_{t\to T}\|u(\cdot,t)\|_\infty=+\infty$,

where $\|u(\cdot,t)\|_\infty=\max_{0\le x\le 1}|u(x,t)|$.

In this case, we say that the solution u blows up in a finite time and the time T is called the blow-up time of the solution u.

The theoretical study of the phenomenon of blow-up has been the subject of investigations of many authors (see [3], [9], [12], [15], [17], [22] and the references cited therein).

Corresponding author:* **Yekre B
Université Nangui Abrogoua, UFR-SFA, Département de Mathématiques et Informatiques, (Côte d'Ivoire)

In particular, in [12], the authors have shown that if γ tends to zero, the above problem has been studied and existence and uniqueness of a classical solution has been proved. Under some assumptions, it is also shown that the classical solution blows up in a finite time and its blow-up time has been estimed.

In this paper, we are investing in the numerical study of the above problem. Let I be a positive integer, where $h=\frac{1}{I}$ is the mesh parameter and define the grid $x_i = ih,\ 0 \le i \le I$ or $x_{i+1} = x_i + h$ and $\Delta t_n = x_{i+1} - x_i$ approximate the solution u of (1)-(3) by the solution $U_h(t)=(U_0(t),U_1(t),\ldots,U_I(t))^T$ of the following semi discrete equations

$$\frac{d}{dt}U_i(t)-\gamma\delta^2U_i(t)=f(U_i(t)),\quad 1\le i\le I,\ t\in(0,T_b^h), \qquad (4)$$

$$U_0(t)=0,\ \ U_I(t)\ge 0, \qquad t\in(0,T_b^h), \qquad (5)$$

$$U_i(0)=\varphi_i\ge 0, \qquad 1\le i\le I, \qquad (6)$$

Where

$$\varphi_{i+1}\ge\varphi_i,\quad 1\le i\le I-1,$$

$$\delta^2U_i(t)=\frac{U_{i+1}(t)-2U_i(t)+U_{i-1}(t)}{h^2},\quad 1\le i\le I-1.$$

Here $(0,T_b^h)$ is the maximal time interval on which $\|U_h\|_\infty<+\infty$ with $\|U_h\|_\infty=\max_{0\le i\le I}\|U_i(t)\|$

When T_b^h is finite we say that the solution $U_h(t)$ exists globally if not, we say that the solution $U_h(t)$ of (4)-(6) blows up on a finite time and the time T_b^h is called the blow-up time of the Solution $U_h(t)$.

In this paper we are interesting in the numerical study of the above problem. Firstly, we show that the solution of a semi-discrete form of (1)-(3) blows up in a finite time when γ is small enough in addition, we prove that the semi discrete blow-up time tends to the real one as γ goes to zero. In the case where the blow-up occurs, we show that the semi discrete blow-up time converges to the real one when the mesh size goes to zero.

Our work was motived by the paper in [1],[2],[9],[15],[19]. In [1] the authors have considered the problem (1)-(3) in the case where the parameter γ equals one. They have prove that the solution of the semi discrete scheme (4)-(6) blows up in finite time and its semi discrete blow-up time converges to real one when the mesh size goes to zero in the case where the initial data, is symetric an large enough. Let us notice that in the case where γ =1 we have shown for our problem that the semi discrete solution exists globally, and is bounded from above by one (see theorem, below...). In [19], the author has shown that the solution of a discrete form of the $u_t(x,t)=u_{xx}(x,t)+u^2(x,t)$ with dirichlet boundary conditions and large initial data blows up in a finite time converges to the real one when the mesh size goes to zero. In [2], semi discrete and discrete schemes have been used to study phenomenon of extinction (we say that a solution extincts in finite time if it reaches the values zero in a finite time).\\Our paper is written in the following manner. In the next section, we give some results about the discrete maximum principle, we give some results which will be used later, in the third section, under some assumptions, we show for small diffusion, the solution of (4)-(6) blows-up time converges to one when the parameter γ tends to zeros we also show that for large diffusion the solution of (4)-(6) exist globally and is bounded from above, in the fourth section, we prove that the case where blow-up occurs, the semi discerete blow up time convergences to the real one when the mesh size goes to zero. Finally, in the last section, we give some numerical results to illustrate our analysis.

Properties of semi discrete problem

In this section, we give some lemmas which will be used later, we prove some results about the semi discrete maximum principle and reveal certains properties concerning the operator δ^2.

Lemma 2.1

Let $a_h\in C^0([0,T],\square^{I+1})$ *and let* $V_h(t)\in C^1([0,T],\square^{I+1})$ *such that*

$$\frac{d}{dt}V_i(T)-\gamma\delta^2V_i(t)+a_i(t)V_i(t)\ge 0,\quad 1\le i\le I-1,\ \ t\in(0,T) \qquad (7)$$

$$V_0(t)\ge 0,\quad V_I(t)\ge 0,\quad t\in(0,T) \qquad .(8)$$

$$V_i(0)\ge 0,\quad 0\le i\le I \qquad (9)$$

then we have $V_i(t)\ge 0,\ 0\le i\le I,\ t\in(0,T)$.

Proof

Let $T_0<T$ *and let* $m=\min_{0\le i\le I,0\le t\le T}V_i(t)$. *Since for* $i\in 0,\ldots,I,\ V_i(t)$ *is continuous function, there exist* $t_0\in[0,T_0]$ *such that* $m=V_{i0}(t_0)$ *for* $i_0\in 0,\ldots,I$. *If* $i_0=0$ *or* $i_0=I$*,we have* $m\ge 0$.
For $i_0\in 1,\ldots,I-1$ *it is not hard to see that*

$$\frac{d}{dt}V_{i0}(t_0)=\lim_{h\to 0}\frac{V_{i0}(t_0)-V_{i0}(t_0-h)}{h}\le 0 \qquad (10)$$

$$\delta^2V_{i0}(t_0)=\frac{V_{i0+1}(t_0)-2V_{i0}(t_0)+V_{i0-1}(t_0)}{h^2}\ge 0 \qquad (11)$$

Define the vector $Z_h(t)=e^{\lambda t}V_h(t)$ *where*
λ *is large enough that* $a_{i0}(t_0)-\lambda>0$. A straighforward computer reveals that

$$\frac{dZ_{i0}(t_0)}{dt}-\gamma\delta^2Z_{i0}(t_0)+(a_{i0}(t_0)-\lambda)Z_{i0}(t_0)\ge 0 \qquad (12)$$

We observe from (10)-(11) that $\frac{dZ_{i_0}(t_0)}{dt}\le 0$ *and delt* $a^2Z_{i_0}(t_0)\ge 0.$ Using (12) we arrive at

$(a_{i_0}(t_0)-\lambda)Z_{i_0}(t_0)\geq 0,$ which implies that $Z_{i_0}(t_0)\geq 0.$

Therefore $V_{i_0}(t_0)=m\geq 0$ and we have the desired result. □

Another form of the maximum principle for semi discrete equation is the following comparison lemma.

Lemma 2.2

Let $V_h(t),\ U_h(t)\in C^1([0,T],R^{I+1})$ and $g\in C^0(R\times R,R)$ such that for $t\in(0,T)$:

$$\frac{dV_i(t)}{dt}-\gamma\delta^2V_i(t)+g(V_i(t),t)<\frac{dU_i(t)}{dt}-\gamma\delta^2U_i(t)+g(U_i(t),t)\quad 1\leq i\leq I-1,\ (13)$$

$V_0(t)< U_0(t),\ V_I(t)< U_I(t),$ (14)

$V_i(0)<U_i(0),\quad 1\leq i\leq I,$ (15)

then we have

$V_i(t)<U_i(t)\quad 0\leq i\leq I,\ \ t\in(0,T).$

Proof

Define the vector $Z_h(t)=U_h(t)-V_h(t)$. Let t_o *be first* $t>0$ such that $Z_i(t)>0$ for $t\in(0,t_0),\ i=0,...,I$ but $Z_{i_0}(t_0)=0$ for certain condition because of (14).

If $i\in 1,...,I-1,$ we observe that

$$\frac{dZ_{i_0}(t_0)}{dt}=\lim_{k\to 0}\frac{Z_{i_0}(t_0)-Z_{i_0}(t_0-k)}{k}\leq 0,$$

$$\delta^2Z_{i_0}(t_0)=\frac{Z_{i_0+1}(t_0)-2Z_{i_0}(t_0)+Z_{i_0-1}(t_0)}{h^2}\geq 0,$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt}-\gamma\delta^2Z_{i_0}(t_0)+fg(U_{i_0}(t_0),t_0)-g(V_{i_0}(t_0),t_0)\leq 0.$$

But this inequality contradicts (13).

The lemma below is a discrete version of the Green′s formula.

Lemma 2.3

Let $V_h(t)$ and $U_h(t)$ two vectors such that $U_0(t)=0$, $U_I(t)=0$, $V_0(t)=0$, $V_I(t)=0$. Then we have

$$\sum_{i=1}^{I-1}hU_i\delta^2V_i=\sum_{i=1}^{I-1}hV_i\delta^2U_i.$$

Proof:

A routine calculation yields

$$\sum_{i=1}^{I-1}hU_i\delta^2V_i=\sum_{i=1}^{I-1}hV_i\delta^2U_i+\frac{V_IU_{I-1}-U_IV_{I-1}+V_0U_1-U_0V_1}{h}$$

and the result follows using the assumptions of the lemma□

The lemma below give us a property of the operator δ^2.

Lemma 2.4

Let $U_h(t)$ be the solution of (4)-(6). Then we have $U_i(t)>0,\quad 1\leq i\leq I-1,\ t\in(0,T_h^b).$

Proof

From lemma2.1, $U_h(t)\geq 0,$ for $t\in(0,T_h^b).$

Assume that there exist a time $t_0\in(0,T_h)$ such that $U_{i_0}(t_0)=0$ for a certain $i_0\in 1,...,I-1,$ we observe that

$$\frac{dU_{i_0}(t_0)}{dt}=\lim_{k\to 0}\frac{U_{i_0}(t_0)-U_{i_0}(t_0-k)}{k}\leq 0$$

$$\delta^2U_{i_0}(t_0)=\frac{U_{i_0+1}(t_0)-2U_{i_0}(t_0)+U_{i_0-1}(t_0)}{h^2}\geq 0,$$

We deduce that

$$\frac{dU_{i_0}(t_0)}{dt}-\gamma\delta^2U_{i_0}(t_0)<0.$$

But this contradicts (4) and we have the desired result.

The following lemma reveals property of the semi discrete solution.

Lemma 2.5

Let $U_h\in R^{I+1}$ such that $U_h\geq 0$ then we have $\delta^2f(U_i)\geq f'(U_i)\delta^2U_i,\quad 1\leq i\leq I-1.$

Proof

Apply Taylor's expension to obtain

$$\delta^2f(U_i)=f'(U_i)\delta^2U_i(t)+\frac{(U_{i+1}-U_i)^2}{2h^2}f''(\theta_i)+\frac{(U_{i+1}-U_i)^2}{2h^2}f''(\eta_i),\quad 1\leq i\leq I,$$

where θ_i is an intermediate value between U_i and U_{i+1} and η_i is the one between U_{i-1} and U_i.

Use the fact $U_h>0$ to complete the rest of the proof□

The lemme below shows that if $U_h(t)$ is the solution of the semidiscrete problem, then $\frac{d}{dt}U_i(t)$ is positive when i is between 1 and $I-1$.

Lemma 2.6

Let $U_h(t)$ be the solution of (4)-(6). Then we have $\frac{d}{dt}U_i(t)>0,\quad 1\leq i\leq I-1.$

Proof

Setting $W_i(t)=\frac{d}{dt}U_i(t),\ 1\leq i\leq I-1,$ it is not hard to see that

$\frac{d}{dt}W_i(t)=\gamma\delta^2W_i(t)+f'(U_i(t))W_i(t),$ (16)

$W_0(t)=0, W_I(t)=0,\quad t\in(0,T_b^h),$ (17)

$W_i(0)\geq 0,\quad 1\leq i\leq I-1.$ (18)

Let t_0 be the first t>0 such that $W_{i_0}(t_0)=0$ for a certain $i_0\in 1,...,I-1$ without loss of generality, we may suppose that i_0 is the smallest i_0 which ensure the equality, we get

$$\frac{dW_{i_0}(t_0)}{dt}=\lim_{k\to 0}\frac{W_{i_0}(t_0)-W_{i_0}(t_0-k)}{k}\leq 0$$

$$\delta^2W_{i_0}(t_0)=\frac{W_{i_0+1}(t_0)-2W_{i_0}(t_0)+W_{i_0-1}(t_0)}{h^2}\geq 0,$$

which guarantees that

$$\frac{dW_{i_0}(t_0)}{dt}-\gamma\delta^2W_{i_0}(t_0)-f(U_{i_0}(t_0))W_{i_0(t_0)}<0.$$

Therefore, we have a contradiction because of (16). The following lemma shows that the solution $U_h(t)$ of the semidiscrete problem is symetric and $\delta^+(U_i(t))$is positive when i is between 0 and $[\frac{I}{2}]-1$. $[\frac{I}{2}]$ is the integer part of the number $\frac{I}{2}$.

Lemma 2.7

Let U_h be the solution of (4)-(6). Then we have for $t\in(0,T_b^h)$

$U_{I-i}(t)=U_i(t),\quad 1\leq i\leq I,$

$\delta^+(U_i(t))\geq 0,\quad 0\leq i\leq[\frac{I}{2}]-1,$ for $t\in(0,T_b^h)$

Proof

Introduce the vector V_h defined as follows $V_i(t)=U_{I-i}(t)$ for $0\le i\le I$.
It is not hard to see that $V_h(t)$$ is a solution of (4)-(6). Now, define the vector Z_h such that $Z_h(t)=U_h(t)-V_h(t)$ It is not hard to see that

$$\frac{d}{dt}Z_i(t)=\gamma\delta^2 Z_i(t)+f(\xi_i(t))Z_i(t),\quad 1\le i\le I-1, \qquad (19)$$

$$Z_0(t)=0,\qquad Z_I(t)=0, \qquad (20)$$

$$Z_i(0)=\varphi_i. \qquad (21)$$

Where $\xi_i(t)$ is an intermediate value between $U_i(t)$ and $V_i(t)$. It follows from lemma2.1 that $V_h(t)=U_h(t)$.Now let us prove that the second part of lemma. Since $U_i(t)>0$ for $t\in(0,T_b^h)$, we observe that $\delta^+U_0(t)>0$.Let t_0 be the first t>0 such that $\delta^+U_i(t)>0$ for $t\in(0,t_0)$ but $\delta^+U_{i_0}(t_0)=0$ for a certain i_0 which is between 0and $[\frac{I}{2}]$-1. As seen yet, $i_0\ge 0$ without loss of the generalinty, we assume that i_0 is the smallest integer which guarantees the quality. Setting,

$$\frac{dZ_{i_0}(t_0)}{dt}=\lim_{k\to 0}\frac{Z_{i_0}(t_0)-Z_{i_0}(t_0-k)}{k}\le 0,$$

$$\delta^2 Z_{i_0}(t_0)=\frac{Z_{i_0+1}(t_0)-2Z_{i_0}(t_0)+Z_{i_0-1}(t_0)}{h^2}\ge 0, if\quad 1\le i_0\le[\frac{I}{2}]-2$$

and we know that if $i_0=[\frac{I}{2}]-1$,

$$\delta^2 Z_{[\frac{I}{2}]-1}(t_0)=\delta^2 U_{[\frac{I}{2}]}(t_0)-\delta^2 U_{[\frac{I}{2}]-1}(t_0)=$$

$$\frac{U_{[\frac{I}{2}]+1}(t_0)-2U_{[\frac{I}{2}]}(t_0)+U_{[\frac{I}{2}]-1}(t_0)-U_{[\frac{I}{2}]}(t_0)+2U_{[\frac{I}{2}]-1}(t_0)-U_{[\frac{I}{2}]-2}(t_0)}{h^2}$$

Using the fact that the discrete solution is symetric, we have either $U_{[\frac{I}{2}]+1}(t)=U_{[\frac{I}{2}]-1}(t)$ or $U_{[\frac{I}{2}]+1}(t)=U_{[\frac{I}{2}]}(t)$ in both cases we find that

$$\delta^2 Z_{[\frac{I}{2}]-1}(t_0)=\frac{Z_{[\frac{I}{2}]-2}(t_0)}{h^2}>0.$$

the above inequality imply that

$$\frac{d}{dt}Z_{i_0}(t_0)-\gamma^2 Z_{i_0}(t_0)+f(\xi_i(t_0))Z_{i_0}(t_0)<0,$$

which is contradict becauseof(19)□

Lemma 2.8

Let U_h be the solution of (4)-(6). Then we have for $t\in(0,T_b^h)$
Let U_h and $U_h\in C^1([0,T],R^{I+1})$ if $\delta^+(U_i)\delta^+(V_i)\ge 0$ and $\delta^-(U_i)\delta^-(V_i)\ge 0$
$\delta^2(U_iVi)\ge U_i\delta^2(V_i)+V_i\delta^2(U_i)$,
where $\delta^+(U_i)=\frac{U_{i+1}-U_i}{h}$ and $\delta^-(U_i)=\frac{U_{i-1}-U_i}{h}$.

Proof

A straightfoward computation yields

$$h^2\delta^2(U_iV_i)=U_{i+1}V_{i+1}-2U_iV_i+U_{i-1}V_{i-1}=(U_{i+1}-U_i)(V_{i+1}-V_i)+V_i(U_{i+1}-U_i)+U_i(V_{i+1}-V_i)$$
$$+U_iV_i-2U_iV_i+(U_{i-1}-U_i)(V_{i-1}-V_i)+(U_{i-1}-U_i)V_i+U_i(V_{i-1}-V_i)+U_iV_i.$$

Which implies that

$$\delta^2(U_iV_i)=\delta^+(U_i)\delta^+(V_i)+\delta^-(U_i)\delta^-(V_i)+V_i\delta^2(U_i).$$

Using the assumption of the lemma.We obtien the desired result□

Blow-up in the semi discrete problem

In the section, we choose $f(U)=e^u$ and $U_i(0)=0$, under some assumptions with respect to the parameter γ and the initial data, we show that the solution U_h of (4)-(6) blow-up in a finite time and estimate its blow-up time. Our first result on blow-up is the following the theorem.

Theorem 3.1

Let U_h be the solution of $(4)-(6)$ and suppose that $\gamma<\frac{e}{\alpha_h}$ with
$\gamma_h=\frac{2(1-\cos(\pi h))}{h^2}$. Then the solution U_h blow-up in a finite time T_b^h which is estimated as follows

$$T_b^h\le\frac{e}{e-\alpha_h\gamma}$$

Proof

Since $(0,T_b^h)$ be the maximal time interval on which $\|U_h(t)\|_\infty$ is finite,
our aim is to show that T_b^h is finite and satisfies the above inequality. Introduce the vector w(t) defined as follows

$$w(t)=\sum_{i=1}^{I-1}\tan(\frac{\pi h}{2})\sin(i\pi h)U_i(t)$$

Taking the derivative of w and t and using (4), with $f(U_i(t))=e^{U_i(t)}$,we find that

$$w^{'}=\gamma\sum_{i=1}^{I-1}\tan(\frac{\pi h}{2})\sin(i\pi h)\delta^2U_i(t)+\sum_{i=1}^{I-1}\tan(\frac{\pi h}{2})\sin(i\pi h)e^{U_i(t)}$$

Due to lemma2.3 and the fact that $\delta^2\sin(i\pi h)=-\alpha_h\sin(i\pi h)$, we obtain

$$w^{'}(t)=-\alpha_h\gamma w(t)+\sum_{i=1}^{I-1}\tan(\frac{\pi h}{2})\sin(i\pi h)e^{U_i(t)}.$$

A routine computation reveals that $\sum_{i=1}^{I-1}\tan(\frac{\pi h}{2})\sin(i\pi h)=1$.Therefore, applying Jensen's inequality, we arrive at

$w^{'}(t)\ge-\gamma\alpha_h w(t)+e^{w(t)}$ which implies that

$$w^{'}(t)\ge e^{w(t)}(1-\gamma\alpha_h\frac{w(t)}{e^{w(t)}})$$

Using the fact $\sup_{s\ge 0}\frac{s}{e^s}=\frac{1}{e}$, we find that $w'(t)\ge e^{w(t)}(1-\frac{\gamma\alpha_h}{e})$.

The above estimate may be written as follows $e^{w(t)}dw\ge e^{w(t)}(1-\frac{\gamma\alpha_h}{e})dt$

Integrating this inequality over $(0,T_b^h)$, we get $T_b^h\le\frac{1}{(1-\frac{\gamma\alpha_h}{e})}$

Hence T_b^h is finite and the proof is complete. □

Theorem 3.2

Let U_h be the solution of (4)-(6). Then we have $T_b^h\ge 1$.

Proof:

Let i_0 be such that $U_{i_0}(t)=\|U_h(t)\|_\infty$ we observe that

$$\delta^2U_{i_0}(t)=\frac{U_{i_0+1}(t)-2U_{i_0}(t)+U_{i_0-1}(t)}{h^2}\le 0\ \text{ whicih implies that}$$

$$\frac{dU_{i_0}(t)}{dt}\le e^{U_{i_0}(t)},$$

that is to say $e^{-U_{i_0}(t)}dU_{i_0}$ dt. Integrating this inequality over $(0,T_b^h)$, we get $T_b^h\ge 1$.

Remark 3.1

Theorem 3.1 and 3.2 show that, the solution of the semi discrete as a function of γ and the initial data blows up in a finite time T_b^h bounded from above and below. If the parameter of the diffusion $\gamma$and the initial data goes to zero we see that T_b^h goes to one. It is not hard to see that one is also the blow-up time for the solution of the following differential equation

$$\lambda^{'}(t)=e^{\lambda(t)},\quad \lambda(0)=0,$$

Thus from 3.1 and 3.2, we prove a well-known result for the continuous problem (see[12]).

The following result shows that the solution of the semi discrete problem exists globally for γ sufficiently large.

Theorem 3.3

If $\gamma\ge\frac{e}{8}$ then the solution $U_h(t)$ of (4)-(6) exist globally and we have.
$0\le U_i(t)\le 4ih(1-ih),\quad t>0,\quad 0\le i\le I.$

Proof

Introduce the vector φ_h defined by $\varphi_i=4ih(1-ih)$, $0 \le i \le I$.

We have $\varphi_{[\frac{I}{2}]}(t) \le 1$. It is not hard to see that.

$$\frac{d\varphi_i}{dt} - \gamma\delta^2\varphi_i = 8\gamma \ge e \ge e^{\varphi_{[\frac{I}{2}]}} \ge e^{\varphi_i}, \quad 1 \le i \le I-1.$$

Setting $Z_h(t)=\varphi_h-U_h(t)$, we find.

$$\frac{dZ_i}{dt} - \gamma\delta^2 Z_i - e^{\xi_i(t)}Z_i \ge 0, \quad 1 \le i \le I-1,$$

$Z_0(t)=0$, $Z_I(t)=0$.

From lemma 2.1, we deduce that $Z_h(t) \ge 0$ for $t \in (0,T_b^h)$ that is to say

$0 \le U_i(t) \le 4ih(1-ih), \quad t>0, 0 \le i \le I, t \in (0,T_b^h),$

this implies that $T_b^h=+\infty$ and the proof is complete□

Remark 3.2

The above theorem shows that for large diffusion, the solution of the semi discrete problem is bounded from above by one. We have seen that for large diffusion, the solution of the semi discret problem exists globally and is bounded from above. The following theorem reveals that in this case it approaches its stationary solution as t goes to infinity.

Theorem 3.3

Assume that the solution $U_h(t)$ of (4)-(6) exists globally and is bounded then $U_h(t)$ goes to $V_h(t)$ as t approaches infinity where $V_h(t)$ is the stationary solution of (4)-(6).

Proof

Introduce the vector $W_h(t)$ such that

$W_i(t) = \sum_{j=1}^{I-1} G_{ij}U_j(t)$ *here* $G_{h,k}$ is the discrete Green function defined by

$$G_{ij} = \begin{cases} \frac{1}{2}ih(1-jh) & \text{if} \quad 0 \le i < j \le I \\ \frac{1}{2}jh(1-ih) & \text{if} \quad 0 \le j \le i \le I \end{cases}$$

Taking the derivative of $W_i(t)$ with respect to t and using (4), a straightforward computation reveals that $\frac{d}{dt}W_i(t) = \sum_{j=1}^{I-1} G_{ij}(\gamma\delta^2 U_j(t) + e^{U_j(t)})$ which implies that

$$\frac{dWi(t)}{dt} = -\gamma U_i(t) + \sum_{j=1}^{I-1} G_{ij}e^{U_j(t)}.$$

From lemma 2.6 $U_j(t)$ is strictly increasing. On the other hand, the last term on the right hand side of the above equality is bounded. Hence, we may conclude that $\frac{dW_i(t)}{dt}$ goes to zero as t approaches infinity. Setting $\lim_{t\to+\infty} U_i(t) = V_i$.

We derive the following equality $V_i = \sum_{j=1}^{I-1} G_{ij}e^{V_j}$ which implies that

$$\gamma\delta^2 V_p + e^{V_i} = 0, \quad 1 \le i \le I-1,$$

$V_0=0$, $V_I=0$, and the proof is complete. □

Convergence of the blow-up time

In this section, under some conditions, we show that the solution of the semi discrete problem blows-up time goes to the real one when the mesh size tends to zero. Firstly, let us prove the convergence of our scheme by the following theorem.

Theorem 4.1

Suppose that $\gamma < \frac{\cos(\frac{\pi h}{2})}{\pi^2}$, then the solution of (4)-(6) blows-up in a finite time T_b^h which satisfies the following estimate $T_b^h \le -\frac{1}{\gamma\pi^2}\ln\left(1-\frac{\gamma\pi^2}{\cos(\frac{\pi h}{2})}\right)$

Proof: Introduce the vector $W_h(t)$ such that

Since $(0,T_b^h)$ is the maximal time interval on which $\|U_h(t)\|_\infty$ is finite, our aim is to show that T_b^h is finite and satisfies the above inequality.

Introduce the vector $J_h(t)$ defined as follows

$$J_i(t) = \frac{dU_i}{dt} - C_i(t)e^{U_i(t)}, \quad 0 \le i \le I.$$

Where $C_i(t)=e^{-\gamma\alpha_h t}\sin(i\pi h)$ with $\alpha_h=2(\frac{1-\cos(\pi h)}{h^2})$. A direct calculation yields

$$J_i(t) = \frac{dU_i}{dt} - C_i(t)e^{U_i(t)}, \quad 0 \le i \le I.$$

$$\frac{dJ_i(t)}{dt} - \gamma\delta^2 J_i(t) = \frac{d}{dt}(\frac{dU_i(t)}{dt} - \gamma\delta^2 U_i(t)) - \frac{dC_i(t)}{dt}e^{U_i(t)} - C_i(t)e^{U_i(t)}\frac{dU_i(t)}{dt} + \gamma\delta^2(C_i(t)e^{U_i(t)}).$$

We observe that $C_h(t)$ is symetric and δ^+C_i is positive for $0 \le i \le [\frac{I}{2}]-1$.

It follows from lemmas 2.5,2.7 and 2.8 that

$$\delta^2(C_ie^{U_i}) \ge C_ie^{U_i}\delta^2U_i + e^{U_i}\delta^2C_i.$$

Use this inequality and the fact that $\frac{dC_i}{dt}-\gamma\delta^2C_i=0$ to obtain

$$\frac{dJ_i}{dt} - \gamma\delta^2 J_i \ge \frac{d}{dt}(\frac{dU_i}{dt} - \gamma\delta^2 U_i) - C_ie^{U_i}(\frac{dU_i}{dt} - \gamma\delta^2 U_i).$$

Taking in to account(4), we arrive at $\frac{dJ_i}{dt} - \gamma\delta^2 J_i \ge C_ie^{U_i}J_i, \quad 1 \le i \le I-1, t \in (0,T_b^h).$

Obviously, $J_0(t)=0$, $J_I(t)=0$ and $J_h(0) \ge 0$, which applying lemma2.1,

we get $J_h(t) \ge 0$, which implies that

$\frac{dU_i}{dt} \ge \sin(i\pi h)e^{-\gamma\alpha_h}e^{U_i}, \quad 1 \le i \le I.$ It is not hard to see that $\alpha_h \le \pi^2$ and

$\sin(\frac{[\frac{I}{2}]\pi h}{2}) \ge \cos(\frac{\pi h}{2})$. We deduce that $\frac{dU_k}{dt} \ge \cos(\frac{\pi h}{2})e^{-\pi^2 t}e^{U_k}$, which implies that

$$e^{-U_k}dU_k \ge \cos(\frac{\pi h}{2})e^{-\pi^2 t}dt, \tag{22}$$

where $k=[\frac{I}{2}]$. Intergrating the above inequality over $(0,T_b^h)$,

we arrive at $\cos(\frac{\pi h}{2})\frac{1-e^{\gamma\pi^2 T_b^h}}{\gamma\pi^2} \le 1$, which implies that $e^{-\gamma\pi^2 T_b^h} \ge 1-\frac{\gamma\pi^2}{\cos(\frac{\pi h}{2})}$,

since $\gamma < \frac{\cos(\frac{\pi h}{2})}{\pi^2}$, then $1-\frac{\gamma\pi^2}{\cos(\frac{\pi h}{2})} > 0$, hence $T_b^h \le -\frac{1}{\gamma\pi^2}ln(1-\frac{\gamma\pi^2}{\cos(\frac{\pi h}{2})})$.

This implies that T_b^h is finite and we have the desired result□

Remark 4.1

Integrating the inequality (22) over (t_0,T_b^h), and using the fact that $\|U_h(t)\|_\infty=U_k(t)$, we get.

$T_b^h - t_0 \le \frac{-1}{\gamma\pi^2}\ln(1-\frac{\gamma\pi^2}{\cos(\frac{\pi h}{2})}e^{\gamma\pi^2 t_0}e^{-\|U_h(t_0)\|_\infty})$. Since $\cos(\frac{\pi h}{2}) \ge \frac{1}{2}$, we deduce that

$$T_b^h - t_0 \le \frac{-1}{\gamma\pi^2}\ln(1-2\gamma\pi^2 e^{\gamma\pi^2 t_0}e^{-\|U_h(t_0)\|_\infty}) \quad for\ t \in (0,T_b^h.$$

The proof of the above theorem allows us to establish the estimation of Remark 4.1 which is crucial to prove the convergence of the semi discrete blow-up time. Unfortunately, the result of Theorem 4.1 is not optimal to determine the semi discrete blow-up solutions because of the restriction on the parameter of diffusion γ. Theorem 3.1 is more acceptable to have semidiscrete blow-up solutions. In order to prove the convergence of the semidiscrete blow-up time, we need to show that the semidiscrete scheme converges, and state the result on the convergence of the scheme by the following

Theorem 4.2

Proof

Since $u \in C^{4,1}$, there exist two positive constant K and M such that

$$\frac{\| U_{xxxx} \|_\infty}{12} \le K, \| U \|_\infty \le K, e^{K+1} \le M. \qquad (23)$$

The problem(4)–(6)has for each h, a unique solution $U_h \in C^1([0,T_b^h], R^{I+1})$.

Let t(h) the greatest value of $t>0$ such that

$$\| U_k(t) - u_k(t) \|_\infty < 1, \quad for\ t \in (0, t(h)). \qquad (24)$$

Since the value of the term on the left hand side of the above inequality is null when t is equal to zero, we deduce that t(h)>0 such for h sufficiently small.

Let $t^*(h)=\min(t(h), T)$. By the triangle inequality, we obtain

$$\| U_k(t) \|_\infty \le \| U(x,t) \|_\infty + \| U_k(t) - u_k(t) \|_\infty, \quad for\ t \in (0, t^*(h))$$

which implies that

$$\| U_k(t) \|_\infty \le 1 + K, \quad for\ t \in (0, t^*(h)). \qquad (25)$$

Let $e_k(t) = U_k(t) - u_k(x,t)$ be the error of discretization.

Using Taylor's expansion, we have,

$$\frac{d}{dt} e_i(t) - \gamma^2 e_i(t) = \frac{h^2}{12} U_{xxxx}(\tilde{x}_i, t) + e^{\xi_i(t)} e_i(t), \quad 1 \le i \le I-1, t \in (0, t^*(h)).$$

Where ξ_i is an intermediate value between $U_i(t)$ and $u(x_i,t)$.

Using (23) and (25), we arrive at

$$\frac{d}{dt} e_i(t) - \gamma\delta^2 e_i(t) \le M |e_i(t)| + Kh^2, \quad 1 \le i \le I-1,\ t \in (0, t^{*(k)}).$$

Let Z_k the vector defined by

$$Z_i = e^{(M+1)t} (\| U_k^0 - u_k(0) \|_\infty + Kh^2), \quad 0 \le i \le I, t \in (0, t^{*(k)}).$$

A direct calculation yields

$$\frac{d}{dt} Z_i - \gamma\delta^2 Z_i > M |Z_i(t)| + Kh^2, \quad 1 \le i \le I-1, t \in (0, t^*(h)),$$

$$Z_0 > e_0, \quad Z_I > e_I, \quad Z_i(0) > e_i(0), \quad 0 \le i \le I.$$

It follows from lemme2.2 that $Z_i > e_i(t)$ for $t \in (0, t^*(h)$, $0 \le i \le I$. By the same way, we also prove that $Z_i > -e_i(t)$ for $t \in (0, t^*(h))$, $0 \le i \le I$, which implies that

$$\| U_k(t) - u_k(t) \|_\infty \le e^{(M+1)t} (\| U_k^0 - u_k(0) \|_\infty + Kh^2), \quad t \in (0, t^*(h)),$$

Let us show that $t^*(h)=T$. Suppose that $T>t(h)$. From (24), we obtain

$$1 = \| U_k(t) - u_k(t) \|_\infty \le e^{(M+1)t} \| U_k^0 - u_k(0) \|_\infty + Kh^2), \quad t \in (0, t^*(h)).$$

Since the term on the right hande side of the above inequality goes to zero as h tends to zero, we deduce that $1 \le 0$, which is impossible. Consequently $t^*(h)=T$, and we obtain the desired result□

Now,we are in a position to prove our main theorem of the section.

Theorem 4.3

Suppose that the problem(1)-(3) has a solution u which blows-up in a finite time T_b such that $u \in C^{4,1}([0,1] \times [0,I])$. Under the assumption of the theorem4.1, the problem (4)-(6) has a solution $U_h(t)$ which blow-up in a finite time T_b^h and the following relation holds $\lim_{h \to 0} T_b^h = T_b$.

Proof

There exists a positive constant N such that

$$\frac{-1}{\gamma\pi^2} \ln(1 - 2\gamma\pi^2 e^{\gamma\pi^2 T_b} e^{-y}) < \frac{\eta}{2}, \text{foryÎ[N,+¥)}.$$

Since u blows up at the time T_b then there exists a time T_1 such that $|T_1 - T_b| \le \frac{\eta}{2}$ and

$$\| U(x,t) \|_\infty > 2N \ for\ t \in [T_1, T_b].$$

Letting $T_2 = \frac{T_1 + T_b}{2}$, we see that u is bounded on the interval $[0, T_2]$. It follows from theorem4.2 that the problem (4)-(6) has a solution $U_h(t)$ which obeys

$\sup_{t \in [0,T_2]} \| U_h(t) - u_h(t) \|_\infty \le N$. Applying the triangle inequality, we get

$\| U_h(t) \|_\infty \ge \| u_h(t) \|_\infty - \| U_h(t) - u_h(t) \|_\infty$, which leads to $\| U_h(t) \| \le N$

for $t \in [0, T_2]$. From theorem 3.1, $U_h(t)$ blow up at the time T_b^h. We deduce from (26) and Remark4.1 that

$|T_b^h - T_b| < |T_b^h - T_2| + |T_2 - T_b| \le \frac{\eta}{2} + \frac{\eta}{2} = \eta$, and we have the desired result□

Numerical Experiments

In this section, we give some computational experiments to confirm the theory developed in the previous section, we consider the following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n^{\text{å}}} = \gamma \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + e^{U_i^{(n)}}, \quad 1 \le i \le I$$

$$U_0^{(n)} = 0, \quad U_I^{(n)} = 0,$$

$$U_i^0 = \phi_i = 0, \quad 0 \le i \le I.$$

And the following implicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \gamma \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} + e^{U_i^{(n)}}, \quad 1 \le i \le I$$

$$U_0^{(n+1)} = 0, \quad U_I^{(n+1)} = 0,$$

$$U_i^0 = \phi_i = 0, \quad 0 \le i \le I.$$

Where $n \ge 0$, $\Delta t_n = h^2 e^{-\| U_h^{(n)} \|_\infty}$ and $\Delta t_n^* = \min\{\frac{h^2}{2}, \Delta t_n\}$, $t^n = \sum_{j=0}^{n-1} \Delta t$.

The explicit scheme may be written as follows:

$$U_i^{(n+1)} = U_i^{(n)} + \Delta t_n^{\text{å}} (\gamma \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + e^{U_i^{(n)}}),$$

$$U_i^{(n+1)} = U_i^{(n)} + \frac{\gamma\Delta t_n^{\text{å}}}{h^2} U_{i+1}^{(n)} + \frac{-2\gamma\Delta t_n^{\text{å}}}{h^2} U_i^{(n)} + \frac{\gamma\Delta t_n^{\text{å}}}{h^2} U_{i-1}^{(n)} + \Delta t_n^{\text{å}} e^{U_i^{(n)}},$$

$$U_i^{(n+1)} = \frac{\gamma\Delta t_n^{\text{å}}}{h^2} U_{i+1}^{(n)} + \frac{1 - 2\gamma\Delta t_n^{\text{å}}}{h^2} U_i^{(n)} + \frac{\gamma\Delta t_n^{\text{å}}}{h^2} U_{i-1}^{(n)} + \Delta t_n^{\text{å}} e^{U_i^{(n)}},$$

$$U_0^{(n)} = 0, \quad U_I^{(n)} = 0,$$

$$For\ i=1,\ U_1^{(n+1)} = \frac{\gamma\Delta t_n^{\text{å}}}{h^2}(U_2^{(n)}) + \left(1 - \frac{2\gamma\Delta t_n^{\text{å}}}{h^2}\right) U_1^{(n)} + \frac{\gamma\Delta t_n^{\text{å}}}{h^2}(U_0^{(n)}) + \Delta t_n^{\text{å}} e^{U_1^{(n)}} \quad or\ U_0^{(n)} = 0\ then$$

$$For\ i=1,\ U_1^{(n+1)} = \left(1 - \frac{2\gamma\Delta t_n^{\text{å}}}{h^2}\right) U_1^{(n)} + \frac{\gamma\Delta t_n^{\text{å}}}{h^2}(U_2^{(n)}) + \Delta t_n^{\text{å}} e^{U_1^{(n)}},$$

$$For\ i=2,\ U_2^{(n+1)} = \frac{\gamma\Delta t_n^{\text{å}}}{h^2}(U_1^{(n)}) + \left(1 - \frac{2\gamma\Delta t_n^{\text{å}}}{h^2}\right) U_2^{(n)} + \frac{\gamma\Delta t_n^{\text{å}}}{h^2}(U_3^{(n)}) + \Delta t_n^{\text{å}} e^{U_2^{(n)}},$$

$$For\ i=3,\ U_3^{(n+1)} = \frac{\gamma\Delta t_n^{\text{å}}}{h^2}(U_2^{(n)}) + \left(1 - \frac{2\gamma\Delta t_n^{\text{å}}}{h^2}\right) U_3^{(n)} + \frac{\gamma\Delta t_n^{\text{å}}}{h^2}(U_4^{(n)}) + \Delta t_n^{\text{å}} e^{U_3^{(n)}} \ldots$$

$$For\ i=I-1,\ U_{I-1}^{(n+1)} = \frac{\gamma\Delta t_n^{\text{å}}}{h^2}(U_I^{(n)}) + \left(1 - \frac{2\gamma\Delta t_n^{\text{å}}}{h^2}\right) U_{I-1}^{(n)} + \frac{\gamma\Delta t_n^{\text{å}}}{h^2}(U_{I-2}^{(n)}) + \Delta t_n^{\text{å}} e^{U_{I-1}^{(n)}} \quad or\ U_I^{(n)} = 0\ then$$

$$For\ i=I-1,\ U_{I-1}^{(n+1)} = \frac{\gamma\Delta t_n^{\text{å}}}{h^2}(U_{I-2}^{(n)}) + \left(1 - \frac{2\gamma\Delta t_n^{\text{å}}}{h^2}\right) U_{I-1}^{(n)} + \Delta t_n^{\text{å}} e^{U_{I-1}^{(n)}}.$$

Lead us to the linear system below

$U_i^{(n+1)} = A U_i^{(n)} + \left(F^{(n)}\right)_i$ where A is a $I \times I$ tridiagonal matrix defined as follows

$$A = \begin{pmatrix} 1 - \frac{2\gamma\Delta t_n^{\text{å}}}{h^2} & \frac{\gamma\Delta t_n^{\text{å}}}{h^2} & 0 & \cdots & 0 \\ \frac{\gamma\Delta t_n^{\text{å}}}{h^2} & 1 - \frac{2\gamma\Delta t_n^{\text{å}}}{h^2} & \frac{\gamma\Delta t_n^{\text{å}}}{h^2} & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{\gamma\Delta t_n^{\text{å}}}{h^2} \\ 0 & \cdots & 0 & \frac{\Delta t_n^{\text{å}}}{h^2} & 1 - \frac{2\gamma\Delta t_n^{\text{å}}}{h^2} \end{pmatrix}$$

implies that

$$A = \begin{pmatrix} a_0 & b_0 & 0 & \cdots & 0 \\ c_0 & a_0 & b_0 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & b_0 \\ 0 & \cdots & 0 & c_0 & a_0 \end{pmatrix}$$

with $a_0 = 1 - 2\gamma \frac{\Delta t_n^{\text{å}}}{h^2}$, $b_0 = \frac{\gamma\Delta t_n^{\text{å}}}{h^2}$, $i = 1, \ldots, I-2$, $c_0 = \frac{\gamma\Delta t_n^{\text{å}}}{h^2}$, $i = 1, \ldots, I-1$,

$\left(F^{(n)}\right)_i = \Delta t_n^{\text{å}} e^{U_i^{(n)}}$ and A a three-diagonal matrix verifying thefollowing properties:

$$A_{i,i} = 1 - 2\gamma \frac{\Delta t_n^{\text{å}}}{h^2} > 0, \quad 0 \le i \le I$$

$$and\ A_{i-1,i} = \frac{\gamma\Delta t_n^{\text{å}}}{h^2} = A_{i,i+1}, \quad 2 \le i \le I-2 \ \ so\ that\ A_{i,i} \ge \sum_{i \ne j} A_{i,j}$$

It follows that $U_h^{(n)}$ exists for $n \ge 0$. In addition, since $U_h^{(0)}$ is non negative, $U_h^{(n)}$ is also non negative for $n \ge 0$.

According to the implicit scheme, it may be written in the following form

$$U_i^{(n+1)} = U_i^{(n)} + \Delta t_n (\gamma \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} + e^{U_i^{(n)}}),$$

$$U_i^{(n+1)} = U_i^{(n)} + \frac{\gamma\Delta t_n}{h^2} U_{i+1}^{(n+1)} + \frac{-2\gamma\Delta t_n}{h^2} U_i^{(n+1)} + \frac{\gamma\Delta t_n}{h^2} U_{i-1}^{(n+1)} + \Delta t_n e^{U_i^{(n)}}, \text{ or}$$

$$(1+\frac{2\gamma\Delta t_n}{h^2})U_i^{(n+1)} - \frac{\gamma\Delta t_n}{h^2} U_{i+1}^{(n+1)} - \frac{\gamma\Delta t_n}{h^2} U_{i-1}^{(n+1)} = U_i^{(n)} + \Delta t_n e^{U_i^{(n)}}, \text{ or}$$

$$-\frac{\gamma\Delta t_n}{h^2} U_{i-1}^{(n+1)} + (1+\frac{2\gamma\Delta t_n}{h^2})U_i^{(n+1)} - \frac{\gamma\Delta t_n}{h^2} U_{i+1}^{(n+1)} = U_i^{(n)} + \Delta t_n e^{U_i^{(n)}}, \text{ with}$$

$$U_0^{(n+1)} = 0, \quad U_I^{(n+1)} = 0,$$

$$\text{For } i=1, \; -\frac{\gamma\Delta t_n}{h^2} U_0^{(n+1)} + (1+\frac{2\gamma\Delta t_n}{h^2})U_1^{(n+1)} - \frac{\gamma\Delta t_n}{h^2} U_2^{(n+1)} = U_1^{(n)} + \Delta t_n e^{U_1^{(n)}} \text{ or } U_0^{(n+1)} = 0 \text{ then}$$

$$\text{For } i=1, \; (1+\frac{2\gamma\Delta t_n}{h^2})U_1^{(n+1)} - \frac{\gamma\Delta t_n}{h^2} U_2^{(n+1)} = U_1^{(n)} + \Delta t_n e^{U_1^{(n)}}$$

$$\text{For } i=2, \; -\frac{\gamma\Delta t_n}{h^2} U_1^{(n+1)} + (1+\frac{2\gamma\Delta t_n}{h^2})U_2^{(n+1)} - \frac{\gamma\Delta t_n}{h^2} U_3^{(n+1)} = U_2^{(n)} + \Delta t_n e^{U_2^{(n)}}$$

$$\text{For } i=3, \; -\frac{\gamma\Delta t_n}{h^2} U_2^{(n+1)} + (1+\frac{2\gamma\Delta t_n}{h^2})U_3^{(n+1)} - \frac{\gamma\Delta t_n}{h^2} U_4^{(n+1)} = U_3^{(n)} + \Delta t_n e^{U_3^{(n)}} \ldots$$

$$\text{For } i=I-1, \; -\frac{\gamma\Delta t_n}{h^2} U_{I-2}^{(n+1)} + (1+\frac{2\gamma\Delta t_n}{h^2})U_{I-1}^{(n+1)} - \frac{\gamma\Delta t_n}{h^2} U_I^{(n+1)} = U_{I-1}^{(n)} + \Delta t_n e^{U_{I-1}^{(n)}}$$

$$\text{or } U_I^{(n+1)} = 0 \text{ then}$$

$$\text{For } i=I-1, \; -\frac{\gamma\Delta t_n}{h^2} U_{I-2}^{(n+1)} + (1+\frac{2\gamma\Delta t_n}{h^2})U_{I-1}^{(n+1)} = U_{I-1}^{(n)} + \Delta t_n e^{U_{I-1}^{(n)}}$$

lead us to the linear system below $BU_i^{(n+1)}=E$, where B is a I×I tridiagonal matrix defined as follows

$$B = \begin{pmatrix} 1+2\frac{\gamma\Delta t_n}{h^2} & -\frac{\gamma\Delta t_n}{h^2} & 0 & \cdots & 0 \\ -\frac{\gamma\Delta t_n}{h^2} & 1+2\frac{\gamma\Delta t_n}{h^2} & -\frac{\gamma\Delta t_n}{h^2} & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & -\frac{\gamma\Delta t_n}{h^2} \\ 0 & \cdots & 0 & -\frac{\gamma\Delta t_n}{h^2} & 1+2\frac{\gamma\Delta t_n}{h^2} \end{pmatrix}$$

implies that

$$B = \begin{pmatrix} a_0 & b_0 & 0 & \cdots & 0 \\ c_0 & a_0 & b_0 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & b_0 \\ 0 & \cdots & 0 & c_0 & a_0 \end{pmatrix}$$

$$\text{with } a_0 = 1+2\frac{\gamma\Delta t_n}{h^2}, \; b_0 = -\frac{\gamma\Delta t_n}{h^2}, \quad i=1,\ldots,I-2,$$

$$c_0 = -\frac{\gamma\Delta t_n}{h^2}, \quad i=1,\ldots,I-1, \; \left(E^{(n)}\right)_i = U_i^{(n)} + \Delta t_n e^{U_i^{(n)}}$$

and B a three-diagonal matrix verifying the following properties:

$(B)_{i,i} = \left(1+\frac{2\gamma\Delta t_n}{h^2}\right) > 0, \quad 0 \le i \le I$ and $(B)_{i-1,i} = -\frac{\gamma\Delta t_n}{h^2} = (B)_{i,i+1} \ge 0,$

$2 \le i \le I\text{-}2$ so that $(B)_{i,i} \ge \sum_{i'j} |(B)_{i,j}|$. It follows that $U_h^{(n)}$ exists for $n \ge 0$.

In addition, since $U_h^{(0)}$ is nonnegative, $U_h^{(n)}$ is also nonnegative for $n \ge 0$.

We need the following definition.

Definition 5.1

The discrete solution $U_h^{(n)}$ of the explicit or of the implicit scheme blows up in a finite time if $\lim_{n\to+\infty} \| U_h^{(n)} \|_\infty = +\infty$ and the series $\sum_{n=0}^{+\infty} \Delta t_n$ converges.

The quantity $\sum_{n=0}^{+\infty} \Delta t_n$ is called the numerical blow-up time of the solution $U_h^{(n)}$.

In the following tables, in rows, we present the numerical blow-up times, the numbers of iterations n, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, and 128.

We take for the numerical blow-up time $t_n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when $\Delta t_n = | T_{n+1} - T_n | \le 10^{-16}$.

The order s of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Numerical experiments for $f(U_i^{(n)}) = e^{U_i^{(n)}}, \; U_i^{(0)} = \phi_i = 0,$

Here, we take $\Delta t_n = \min\{\frac{h^2}{4N}, \frac{h^2}{e^{\|U_h^{(n)}\|_\infty}}\}$ for the explicit scheme and $\Delta t_n = \frac{h^2}{e^{\|U_h^{(n)}\|_\infty}}$ for the implicit scheme

First case: $\gamma = \frac{1}{10}$

Table 1 Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

I	t^n	n	temps CPU	S
16	1.235838	8442	–	–
32	1.234929	32669	–	–
64	1.234708	125123	1	2.04
128	1.234655	483657	8	2.04
256	1.234642	1851864	72	2.04
512	1.234639	7083689	483	2.03

Table 2 Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

I	t^n	n	temps CPU	S
16	1.238175	8460	1	–
32	1.235513	32595	1	–
64	1.234854	125248	2	2.02
128	1.234691	480080	11	2.01
256	1.234651	1835541	83	2.01
512	1.234641	6999689	625	2.00

Second case: $\gamma = \dfrac{1}{50}$

Table 3 Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

I	t^n	n	temps CPU	S
16	1.004838	8082	–	–
32	1.003633	30920	–	–
64	1.003636	118139	1	1.99
128	1.003627	450231	8	2.00
256	1.003621	1711544	69	2.00
512	1.003820	6490830	446	2.00

Table 4 Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

I	t^n	n	temps CPU	S
16	1.006768	8012	–	–
32	1.004548	30643	–	–
64	1.004003	116966	1	2.00
128	1.003865	445465	10	2.00
256	1.003830	1692245	75	2.00
512	1.003822	6410389	573	2.00

Third case: $\gamma = \dfrac{1}{100}$

Table 5 Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

I	t^n	n	temps CPU	S
16	1.001804	8066	–	–
32	1.000499	30843	–	–
64	1.000179	117686	1	2.01
128	1.000103	448045	8	2.00
256	1.000091	1701423	59	2.00
512	1.000086	6442522	444	2.00

Table 6 Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

I	t^n	n	temps CPU	S
16	1.002217	7988	1	–
32	1.000613	30528	1	–
64	1.000216	116434	2	2.02
128	1.000117	443029	11	2.01
256	1.000093	1681313	75	2.00
512	1.000086	6362082	563	2.00

Fourth case: $\gamma = \dfrac{1}{1000}$

Table 7 Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method.

I	t^n	n	temps CPU	S
16	1.001653	8064	–	–
32	1.000413	30836	1	–
64	1.000103	117662	1	2.00
128	1.000026	447926	8	2.00
256	1.000006	1700844	59	2.00
512	1.000002	6439966	437	2.00

Table 8 Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

I	t^n	n	temps CPU	S
16	1.001954	7986	–	–
32	1.000488	30522	–	–
64	1.000122	116404	1	2.00
128	1.000031	442898	10	2.00
256	1.000008	1680734	74	2.00
512	1.000002	6359526	557	2.00

Remark 5.1

In the case where the initial data is null, $\varphi_i = 0,$ and the reaction term increases as a function of $f(u) = e^u$ it is not hard to see that the blow-up time of the solution goes to one (Tables 1-8) when the value of γ decays to zero as we have shown in Remark 3.1.

In the following, we also give some plots to illustrate our analysis. In Figures 1 to 4, we can appreciate that the discrete solution blows up globally. Let us notice that, theoretically, we know that the continuous solution blows up globally under the assumptions given in the introduction of the present paper.

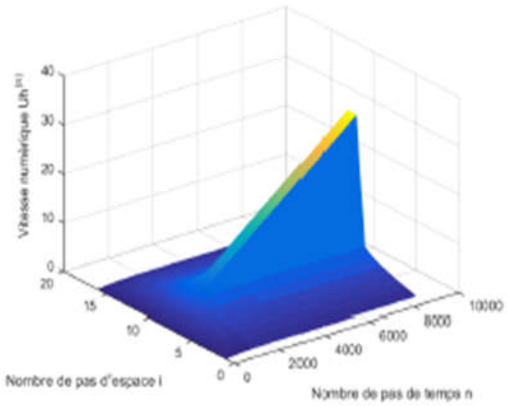

Figure1 :

Evolution of the discrete solution, source

$f(u)=e^u$, γ=1/10, φ_i = 0, I=16 (implicitscheme)

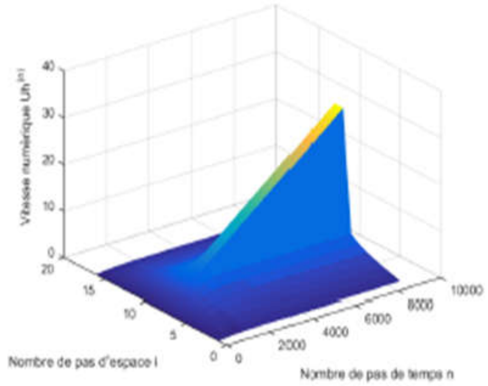

Figure 2 :

Evolution of the discrete solution, source

f(u)=e^{u} , γ=1/10, $\varphi_i = 0$, I=16 (explicite scheme)

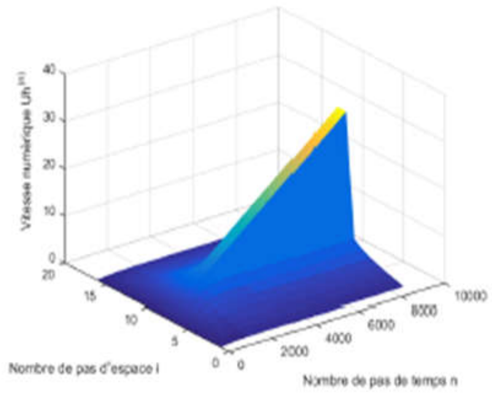

Figure 3 :

Evolution of the discrete solution, source

f(u)=e^{u} , γ=1/50, $\varphi_i = 0$, I=32 (implicit scheme)

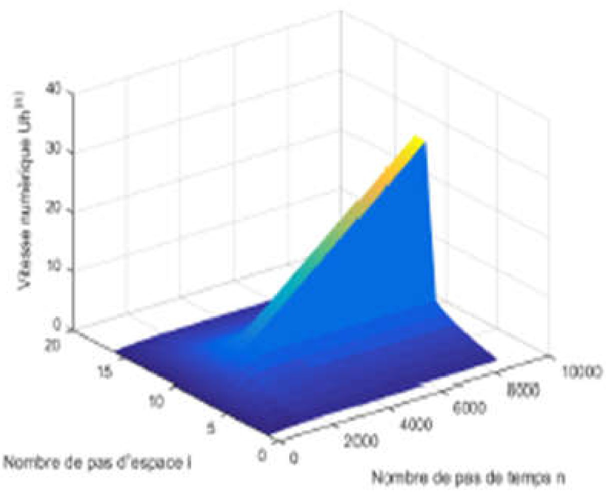

Figure 4 :

Evolution of the discrete solution, source

f(u)=e^{u} , γ=1/50, $\varphi_i = 0$, I=32 (explicite scheme)

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