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STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS AND THEIR PROPERTIES

Pandit U. Chopade

Department of Mathematics, Dnyanopasak College, Jintur-431 509, India

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ABSTRACT

In this paper we discuss the eigenvalue problems which are useful for solving the differential equations to a general class of boundary value problems that share as common set of properties. The Sturm-Liouville Problems define a class of eigenvalue problems, which include many special cases. We also discuss the relation between Strum-Liouville problems and self adjoint problems.

INTRODUCTION

The boundary value problems are commonly associated with the name of Sturm and Liouville. They consists of a differential equation of the form

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0 \qquad (1.1)$$

on the interval $0 < x < 1$, together with the boundary conditions

$$a_1y(0) + a_2y'(0) = 0, b_1y(1) + b_2y'(1) = 0 \qquad (1.2)$$

at the end points. It is often convenient to introduce the linear homogeneous differential operator L defined by

$$L[y] = -[p(x)y']' + q(x)y. \qquad (1.3)$$

Then the differential equation (1.1) can be written as

$$L[y] = \lambda r(x)y = 0. \qquad (1.4)$$

We assume that the functions p, p', q and r are continuous on the interval $0 \leq x \leq 1$ and, further, that $p(x) > 0$ and $r(x) > 0$ at all points in $0 \leq x \leq 1$. These assumptions are necessary to render the theory as simple as possible while retaining considerable generality. It turns out that these conditions are satisfied in many significant problems in mathematical physics. For example, the equation $y'' + \lambda y = 0$ is of the form (1.1) with $p(x) = 1, q(x) = 0,$ and $r(x) = 1$. The boundary conditions (1.2) are said to be separated; that is, each involves only one of the boundary points. These are the most general separated boundary conditions that are possible for second order differential equation. The boundary condition (1.2) is said to be periodic, if

$$y(-L) = 0 = y(L),\ y'(-L) = 0 = y'(L).$$

Properties of Sturm-Liouville BVP

Before proceeding to establish some of the properties of the Sturm-Liouville problem (1.1), (1.2), it is necessary to derive an identity, known as Lagrange's identity, which is basic to the study of linear boundary value problems. Let u and v be functions having continuous second derivatives on the interval $0 \leq x \leq 1$. Then

$$\int_0^1 L[u]v dx = \int_0^1 [-(pu')'v + quv]dx.$$

Integrating the first term on the right side twice by parts, we obtain

$$\int_0^1 L[u]v dx = -p(x)u'(x)v(x)\Big|_0^1 + p(x)v'(x)u(x)\Big|_0^1 + \int_0^1 [-u(pv')' + quv]dx.$$

Corresponding author: **Pandit U. Chopade**
Department of Mathematics, Dnyanopasak College, Jintur-431 509, India

$$= -p(x)[u'(x)v(x) - v'(x)u(x)]|_0^1 + \int_0^1 uL[v]dx.$$

Hence, on transposing the integral on the right side, we have

$$\int_0^1 [L[u]v - uL[v]]dx = -p(x)[u'(x)v(x) - v'(x)u(x)]|_0^1 \quad (2.1)$$

which is Lagrange's identity.

Now let us suppose that the function u and v in (2.1) also satisfy the boundary conditions (1.2). Then, if we assume that $a_2 \neq 0$ and $b_2 \neq 0$, the right side of (2.1) becomes

$$-p(x)[u'(x)v(x) - v'(x)u(x)]|_0^1 = -p(1)[u'(1)v(1) - v'(1)u(1)] + p(0)[u'(0)v(0) - v'(0)u(0)]$$

$$= -p(1)[-\frac{b_1}{b_2}u(1)v(1) + \frac{b_1}{b_2}u(1)v(1)]$$

$$+p(0)[-\frac{a_1}{a_2}u(0)v(0) + \frac{a_1}{a_2}u(0)v(0)]$$

$$= 0.$$

The same result holds if either a_2 or b_2 is zero. Thus, if the differential operator L is defined by (1.3), and if the functions u and v satisfy the boundary conditions (1.2), Lagrange's identity reduces to

$$\int_0^1 [L[u]v - uL[v]]dx = 0 \quad (2.2)$$

Let us now write (2.2) in slightly different way. We introduce the inner product (u, v) of two real-valued functions u and v on the interval $0 \leq x \leq 1$, by

$$(u, v) = \int_0^1 u(x)v(x)dx. \quad (2.3)$$

In this notation (2.2) becomes

$$(L[u], v) - (u, L[v]) = 0. \quad (2.4)$$

In proving Theorem 2.1 below it is necessary to deal with complex-valued functions. We define the inner product of two complex-valued functions on the interval $0 \leq x \leq 1$ as

$$(u, v) = \int_0^1 u(x)\overline{v(x)}dx \quad (2.5)$$

where $\bar{v}$ is the complex conjugate of v. Clearly, (2.5) coincides with (2.3) if u and v are real. It is important to know that (2.4) remains valid under the stated conditions if u and v are complex valued functions and if the inner product (2.5) is used. To see this, one can start with the quantity $\int_0^1 L[u]\bar{v}dx$ and retrace the steps leading to (2.2), making use of the fact that $p(x), q(x), a_1, a_2, b_1, b_2$ are all real quantities.

We now consider some of implications of (2.4) for the Sturm-Liouville boundary value problem (1.1), (1.2). We assume without proof that this problem actually has eigenvalues and eigenfunctions. If the Sturm-Liouville problem (1.1), (1.2) has a non-zero solution $y(x)$ on the interval $0 \leq x \leq 1$, then we say λ is an eigenvalues and that $y(x)$ is corresponding eigenfunction of Sturm-Liouville problem (1.1), (1.2). An eigenvalues of the Sturm-Liouville problem (1.1), (1.2) are said to be simple if to each eigenvalue there correspond only one linearly independent eigenfunction, otherwise eigenvalue is called multiple eigenvalue.

In Theorem 2.1 to 2.4 below, we state several of their important properties. Each property is illustrated by the basic Sturm-Liouville problem

$$y'' + \lambda y = 0, y(0) = 0, y(1) = 0, \quad (2.6)$$

whose eigenvalues are $\lambda_n = n^2\pi^2$, with the corresponding eigenfunctions $\varphi_n(x) = \sin n\pi x$.

Theorem 2.1 All the eigenvalues of the Sturm-Liouville problem (1.1), (1.2) are real.

Proof: To prove this theorem, let us suppose that λ is a positive complex eigenvalue of the problem (1.1), (1.2) and that φ is a corresponding eigenfunction, also possibly complex-valued. Let us write $\lambda = \mu + iv$ and $\varphi(x) = U(x) + iV(x)$, where $\mu, v, U(x)$ and $V(x)$ are real. Then, if we let $u = \varphi$ and also $v = \varphi$ in (2.4), we have

$$(L[\varphi], \varphi) = (\varphi, L[\varphi]). \quad (2.7)$$

However, we know that $L[\varphi] = \lambda r\varphi$, so (2.7) becomes

$$(\lambda r\varphi, \varphi) = (\varphi, \lambda r\varphi). \quad (2.8)$$

Writing out (2.8) in full, using the definition (2.5) of the inner product, we obtain

$$\int_0^1 \lambda r(x)\varphi(x)\overline{\varphi(x)}dx = \int_0^1 \varphi(x)\overline{\lambda r(x)\,\varphi(x)}dx.$$

(2.9) The weight function $r: [0, 1] \to R$, such that $r(x) > 0$, (2.9) reduces to

$$(\lambda - \bar{\lambda}) \int_0^1 r(x)\varphi(x)\overline{\varphi(x)}dx = 0,$$

$$(\lambda - \bar{\lambda}) \int_0^1 r(x)[U^2(x) + V^2(x)]dx = 0. \quad (2.10)$$

The integrand in (2.10) is non-negative and not identically zero. Since the integrand is also continuous, it follows that the integrand is positive. Therefore, the factor $\lambda - \bar{\lambda} = 2iv$ must be zero. Hence $v = 0$ and λ is real, so the theorem is proved.

An important consequence of Theorem 2.1 is that in finding eigenvalues and eigenfunctions. It is also possible to show that the eigenfunctions of the boundary value problem (1.1), (1.2) are real.

Theorem 2.2 (Orthogonality Property) If φ_1 and φ_2 are two eigenfunctions of the Sturm-Liouville problem (1.1), (1.2) corresponding to eigenvalues λ_1 and λ_2, respectively, and if $\lambda_1 \neq \lambda_2$, then

$$\int_0^1 r(x)\varphi_1(x)\varphi_2(x)dx = 0. \quad (2.11)$$

Proof: We note that φ_1 and φ_2 satisfy the differential equations

$$L[\varphi_1] = \lambda_1 r\varphi_1 \quad (2.12)$$

and

$L[\varphi_2] = \lambda_2 r\varphi_2,$ (2.13)

respectively. If we let $u = \varphi_1, v = \varphi_2$, and substitute for $L[u]$ and $L[v]$ in (2.4), we obtain

$(\lambda_1 r\varphi_1,\ \varphi_2) - (\varphi_1,\ \lambda_2 r\varphi_2) = 0,$

(2.5) implies that,

$$\lambda_1 \int_0^1 r(x)\varphi_1(x)\overline{\varphi_2(x)}dx - \lambda_2 \int_0^1 \overline{r(x)}\varphi_1(x)\overline{\varphi_2(x)}dx = 0.$$

Because λ_2, $r(x)$ and $\varphi_2(x)$ are real, this equation becomes

$$(\lambda_1 - \lambda_2)\int_0^1 r(x)\varphi_1(x)\varphi_2(x)dx = 0.$$

Since by hypothesis $\lambda_1 \neq \lambda_2$, it follows that φ_1 and φ_2 must satisfy (2.11), and the theorem is proved.

Theorem 2.3 The eigenvalues of the Sturm-Liouville problem (1.1), (1.2) are all simple. Further, the eigenvalues form an infinite sequence, and can be ordered according to increasing magnitude so that

$$\lambda_1 < \lambda_2 < \cdots \lambda_n < \cdots.$$

Moreover, $\lambda_n \to \infty$ as $n \to \infty$.

Again we note that all the properties stated in Theorem 2.1 to 2.3 are exemplified by eigenvalues $\lambda_n = n^2\pi^2$ and eigenfunctions $\varphi_n(x) = \sin n\pi x$ of the example (2.6). Clearly, the eigenvalues are real. The eigenfunctions satisfy the orthogonality relation

$$\int_0^1 \varphi_m(x)\varphi_n(x)\,dx = \int_0^1 \sin m\pi x\ \sin n\pi x\ dx = 0,$$
$m \neq n$, (2.15)

which establish by direct integration. Further, the eigenvalues can be ordered so that $\lambda_1 < \lambda_2 < \cdots \lambda_n < \cdots$, and $\lambda_n \to \infty$ as $n \to \infty$. Finally, to each eigenvalue there corresponds a single linearly independent eigenfunction.

We will now assume that the eigenvalues of Sturm-Liouville problem (1.1), (1.2) are ordered as indicated in Theorem 2.3. Associated with the eigenvalue λ_n is a corresponding eigenfunction φ_n, determined up to a multiplicative constant. It is often convenient to choose the arbitrary constant multiplying each eigenfunction so as to satisfy the condition

$$\int_0^1 r(x)\,\varphi_n{}^2(x)dx = 1, \quad n = 1, 2, \ldots \qquad (2.16)$$

Equation (2.16) is called a normalization condition, and eigenfunctions satisfying this condition are said to be normalized. Indeed, in this case, the eigenfunctions are said to form an orthonormal set (with respect to the weight function r) since they already satisfy the orthogonality relation (2.11). It is useful to combine (2.11) and (2.16) into a single equation. To this end we introduce the symbol δ_{mn}, known as the Kronecker delta and define by

$$\delta_{mn} = \begin{cases} 0, if\ m \neq n, \\ 1\ if\ m = n. \end{cases} \qquad (2.17)$$

Making use of the Kronecker delta, we can write (2.11) and (2.16) as

$$\int_0^1 r(x)\varphi_m(x)\varphi_n(x)dx = \delta_{mn}. \qquad (2.18)$$

We now turn to the question of expressing a given function *f* as a series of eigenfunctions of the Sturm-Liouville problem (1.1). (1.2).

Now suppose that a given function *f* is continuous and has piecewise continuous derivatives on $0 \leq x \leq 1$, and satisfying the boundary conditions $f(0) = f(1) = 0$, can be expressed in an infinite series of eigenfunctions of Sturm-Liouville problem (1.1), (1.2). If this can be done, then we have

$$f(x) = \sum_{n=1}^{\infty} c_n\varphi_n(x), \qquad (2.19)$$

where the functions $\varphi_n(x)$ satisfy (1.1), (1.2), and the orthogonality condition (2.18). To compute the coefficient in the series (2.19), we multiply equation (2.19) by $r(x)\varphi_m(x)$, where *m* is a fixed positive integer, and integrate from $x = 0$ to $x = 1$. Assuming that the series can be integrated term by term we obtain

$$\int_0^1 r(x)f(x)\varphi_m(x)dx = \sum_{n=1}^{\infty} c_n \int_0^1 r(x)\varphi_m(x)\varphi_n(x)dx = \sum_{n=1}^{\infty} c_n\delta_{mn}. \qquad (2.20)$$

Hence, using the definition of δ_{mn}, we have

$$c_m = \int_0^1 r(x)f(x)\varphi_m(x)dx = (f,\ r\varphi_m), m = 1,2,\ldots. \qquad (2.21)$$

The coefficients in the series (2.19) have thus been formally determined.

Theorem 2.4 Let $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ be the normalized eigenfunctions of the Sturm-Liuoville problem (1.1), (1.2):

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0,$$
$$a_1 y(0) + a_2 y'(0) = 0, b_1 y(1) + b_2 y'(1) = 0.$$

Let *f* and *f'* be piecewise continuous on $0 \leq x \leq 1$. Then the series (2.19) whose coefficient c_m are given by (2.21) converges to $\frac{[f(x+)+f(x-)]}{2}$ at each point in the open interval $0 < x < 1$.

If *f* satisfies further conditions, then a stronger conclusion can be established. Suppose that, in addition to the hypothesis of Theorem 2.4, the function *f* is continuous on $0 \leq x \leq 1$. If $a_2 = 0$ in the first of equation (1.2) [so that $\varphi_n(0) = 0$], then assume that $f(0) = 0$. Similarly, if $b_2 = 0$ in the second of equation (1.2), assume that $f(1) = 0$. Otherwise no boundary conditions need be prescribed for *f*. Then the series (2.19) converges to $f(x)$ at each point in the closed interval $0 \leq x \leq 1$

Self-adjoint problems

Let us consider the boundary value problem consisting of the differential equation

$$L[y] = \lambda r(x)y,\ 0 < x < 1, \qquad (3.1)$$

Where

$$L[y] = P_n(x)\frac{d^n y}{dx^n} + \ldots + P_1(x)\frac{dy}{dx} + P_0(x)y,$$

(3.2) and *n* linear homogeneous boundary conditions at the endpoints. If equation (2.4) is valid for every pair of sufficiently differentiable functions that satisfy the boundary conditions, then the given problem is said self-adjoint. It is important to observe that (2.4) involves restrictions on both the differential equation and boundary conditions. The differential operator *L* must be such that the same operator appears in both terms of (2.4) this require that *L* be of even order. Further, a second order operator must have the form (1.3); a fourth order operator must have the form

$$L[y] = [p(x)y'']'' - [q(x)y']' + s(x)y \qquad (3.3)$$

and higher order operators must have an analogous structure. In addition, the boundary conditions must be such as to eliminate the boundary terms that arise during the integration by parts used in deriving (2.4). For, example, in a second order problem this is true for separated boundary conditions (1.2) and also in certain other problem, one of which is given in Example 3.5.

Let us suppose that we have a self-adjoint boundary value problem for (3.1) where $L[y]$ is given by (3.3). We assume that p, q, r, and s are continuous on $0 \leq x \leq 1$, and that derivatives of p and q indicated in (3.3) are also continuous. If in addition $p(x) > 0$ and $r(x) > 0$ for $0 \leq x \leq 1$ then there is an infinite sequence of real eigenvalues tending to $+\infty$, the eigenfunctions are orthogonal with respect to the weight function r; that is, $r: [0,1] \to R$, such that $r(x) > 0$, for all$x \in [a,b]$, and an arbitrary function can be expressed as a series of eigenfunctions. However, the eigenfunctions may not simple in these more general cases.

Example 3.1 For $\lambda \in R$, solve

$$y'' + \lambda y = 0, y(0) = 0, y'(\pi) = 0. \quad (3.4)$$

Solution:

Case 1. Let $\lambda < 0$. Then $\lambda = -\mu^2$, where μ is a real and non-zero. The general solution of ODE in (3.4) is given by

$$y(x) = Ae^{\mu x} + Be^{-\mu x}. \quad (3.5)$$

This y satisfies boundary conditions in (3.4) if and only if $A = B = 0$. That is $y \equiv 0$. Therefore, there are no negative eigenvalues.

Case 2. Let $\lambda = 0$. In this case, it easily follows that trivial solution is the only solution of

$$y'' = 0, y(0) = 0, y'(\pi) = 0 \quad (3.6)$$

Thus, 0 is not an eigenvalue.

Case 3. $\lambda > 0$. Then $\lambda = \mu^2$, where μ is a real and non-zero. The general solution of ODE in (3.4) is given by

$$y(x) = A\cos\mu x + B\sin\mu x \quad (3.7)$$

This y satisfies boundary conditions in (3.4) if and only if $A = 0$ and $B\cos\mu x = 0$. But $B\cos\mu x = 0$ if and only if, either $B = 0$ or $\cos\mu x = 0$. The condition $A = B = 0$ means $y \equiv 0$. This does not yield any eigenvalue. If $y \neq 0$, then $B \neq 0$. Thus $\cos\mu x = 0$ hold. This last equation has solution given by $\mu = \frac{2n-1}{2}$, for $n = 0, \pm1, \pm2, \ldots$. Thus eigenvalues are given by

$$\lambda_n = \frac{2n-1}{2}, n = 0, \pm1, \pm2, \ldots \quad (3.8)$$

And the corresponding eigenfunctions are given by

$$\varphi_n(x) = B\sin\left(\frac{2n-1}{2}x\right), n = 0, \pm1, \pm2, \ldots$$

Note: All the eigenvalues are positive. The eigenfunctions corresponding to each eigenvalue form a one dimensional vector space and so the eigenfunctions are unique up to a constant multiple.

Example 3.2 For $\lambda \in R$, solve

$$y'' + \lambda y = 0, y(0) - y(\pi) = 0,\ y'(0) - y'(\pi) = 0. \quad (3.9)$$

Solution: This is not a Sturm-Liouville boundary value problem. It is the mixed boundary condition unlike the separated boundary condition above.

Case 1. Let $\lambda < 0$. Then $\lambda = -\mu^2$, where μ is a real and non-zero. In this case it is easily verified that trivial solution is the only solution of (3.9).

Case 2. Let $\lambda = 0$. In this case, it easily follows that solution of (3.9) is given by

$$y(x) = A + Bx. \quad (3.10)$$

This y satisfies boundary conditions in (3.9) if and only if $B = 0$. Thus A remains arbitrary. Thus 0 is an eigenvalue with eigenfunction being any non-zero constant. Note that eigenvalue is simple.

Case 3. $\lambda > 0$. Then $\lambda = \mu^2$, where μ is a real and non-zero. The general solution of ODE in (3.9) is given by

$$y(x) = A\cos\mu x + B\sin\mu x \quad (3.11)$$

This y satisfies boundary conditions in (3.5) if and only if

$$A\sin\mu x + B(1 - \cos\mu x) = 0,$$
$$A(1 - \cos\mu x) - B\sin\mu x = 0.$$

This has non-trivial solution for the pair (A, B) iff

$$\begin{vmatrix} \sin\mu x & (1-\cos\mu x) \\ (1-\cos\mu x) & -\sin\mu x \end{vmatrix} = 0.$$

That is $\cos\mu x = 1$. This implies that $\mu = \pm 2n, n \in N$, and hence $\lambda = 4n^2, n \in N$. Thus positive eigenvalues are given by

$$\lambda_n = 4n^2, n \in N$$

and the eigenfunctions corresponding to λ_n are given by

$$\varphi_n(x) = \cos 2nx, \phi_n(x) = \sin 2nx, n \in N.$$

Note: All the eigenvalues non negative. There are two linearly independent eigenfunctions namely $\cos 2nx$ and $\sin 2nx$ corresponding to each positive eigenvalue$\lambda_n = 4n^2$.

Example 3.3 Determine the normalized eigenfunction of

$$y'' + \lambda y = 0, y(0) = 0, y(1) = 0.$$

Solution: The eigenvalues of this BVP are

$$\lambda_n = n^2\pi^2,\ n = 1, 2, \ldots,$$

and the corresponding eigenfunctions are

$$\varphi_n(x) = k_n sin(n\pi x),$$

respectively. In this case the weight function is $r(x) = 1$. To satisfy equation (2.16) we must choose k_n so that

$$\int_0^1 (k_n sin(n\pi x))^2 dx = 1, \quad n = 1,2,\ldots. \quad (3.14)$$

Since

$$k_n^2 \int_0^1 sin^2 n\pi x\, dx = k_n^2 \int_0^1 \left(\frac{1}{2} - \frac{1}{2}\cos 2n\pi x\right) dx = \frac{1}{2}k_n^2, n = 1, 2, \ldots \quad (3.14)$$

is satisfied if $k_n = \sqrt{2}, n = 1, 2, \ldots$. Hence the normalized eigenfunctions of the given boundary value problem are

$$\varphi_n(x) = \sqrt{2} sin(n\pi x), n = 1, 2, \ldots$$

Example 3.4 Determine the normalized eigenfunctions of the boundary value problem

$y'' + \lambda y = 0, y(0) = 0, y'(1) + y(1) = 0.$

Solution: We observe that the eigenvalues of given BVP, λ_n satisfy the equation

$$\sin\sqrt{\lambda_n} + \sqrt{\lambda_n}\ cos\sqrt{\lambda_n} = 0, n = 1, 2, \dots, \quad (3.15)$$

and that corresponding eigenfunctions are

$$\varphi_n(x) = k_n sin(\sqrt{\lambda_n}\, x), n = 1, 2, \dots \quad (3.16)$$

where k_n is arbitrary. We determine k_n from the normalized condition (2.16). Since $r(x) = 1$ in this problem, we have $\int_0^1 \varphi_n^2(x)dx = k_n^2 \int_0^1 sin^2(\sqrt{\lambda_n}\, x)\, dx$

$$= k_n^2 \int_0^1 \left(\frac{1}{2} - \frac{1}{2}\cos 2\sqrt{\lambda_n}\, x\right) dx$$

$$= k_n^2 \left(\frac{x}{2} - \frac{sin(2\sqrt{\lambda_n}\, x)}{4\sqrt{\lambda_n}}\right) \Big|_0^1$$

$$= k_n^2 \left(\frac{2\sqrt{\lambda_n} - sin(2\sqrt{\lambda_n})}{4\sqrt{\lambda_n}}\right)$$

$$= k_n^2 \left(\frac{\sqrt{\lambda_n} - sin\sqrt{\lambda_n}.\cos\sqrt{\lambda_n}}{2\sqrt{\lambda_n}}\right)$$

$$= k_n^2 \left(\frac{1 + \cos^2\sqrt{\lambda_n}}{2}\right),$$

where in the last step we have used (3.15). Hence, to normalize the eigenfunctions. φ_n, we must choose

$$k_n = \left(\frac{2}{1 + \cos^2\sqrt{\lambda_n}}\right)^{\frac{1}{2}}, n = 1, 2, \dots.$$

The normalized eigenfunctions of the given BVP are

$$\varphi_n(x) = \frac{\sqrt{2}\, sin\sqrt{\lambda_n}\, x}{(1 + \cos^2\sqrt{\lambda_n})^{\frac{1}{2}}}; n = 1, 2, \dots.$$

We turn now to the relation between Sturm-Liouville problem and Fourier series.

Example 3.5 Find the eigenvalues and eigenfunctions of the boundary value problem

$$y'' + \lambda y = 0, \quad (3.18)$$
$$y(-L) - y(L) = 0,\ y'(-L) - y'(L) = 0. \quad (3.19)$$

Solution: This is not Sturm-Liouville problem because the boundary conditions are not separated. The boundary conditions (3.19) are periodic. Nevertheless, it is straightforward to show that the problem (3.19), (3.20) is self-adjoint. A simple calculation establishes that $\lambda_0 = 0$ is an eigenvalues and that the corresponding eigenfunction is $\varphi_n(x) = 1$. Further, there are additional eigenvalues

$$\lambda_1 = (\tfrac{\pi}{L})^2,\ \lambda_1 = (\tfrac{2\pi}{L})^2,\ \dots, \lambda_n = (\tfrac{n\pi}{L})^2, \dots.$$

To each of these non-zero eigenvalues there corresponds two linearly independent eigenfunctions; for example, corresponding to λ_n are the two eigenfunctions

$$\varphi_n(x) = \cos\frac{n\pi x}{L}, \phi_n(x) = \sin\frac{n\pi x}{L}, n \in N.$$

This illustrate that the eigenvalues may not be simple when the boundary conditions are not separated. Further, if we seek to expand a given function f of period $2L$ in a series of eigenfunctions of the problem (3.18), (3.19). we obtain the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

which is the Fourier series for f.

Eigenvalue Problem Summary

The Dirichlet Problem

$$\left.\begin{matrix} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X(L) \end{matrix}\right\} \Rightarrow \begin{cases} \lambda_n = \dfrac{n\pi}{L}, & n = 1, 2, \dots \\ X_n(x) = \sin\left(\dfrac{n\pi}{L} x\right) \end{cases}$$

The Neumann Problem

$$\left.\begin{matrix} X'' + \lambda^2 X = 0 \\ X'(0) = 0 = X'(L) \end{matrix}\right\} \Rightarrow \begin{cases} \lambda_n = \dfrac{n\pi}{L}, & n = 1, 2, \dots \\ X_n(x) = \cos\left(\dfrac{n\pi}{L} x\right) \end{cases}$$

The Periodic Boundary Value Problem

$$\left.\begin{matrix} X'' + \lambda^2 X = 0 \\ X(-L) = 0 = X(L) \\ X'(-L) = 0 = X'(L) \end{matrix}\right\} \Rightarrow \begin{cases} \lambda_n = \dfrac{n\pi}{L}, \quad n = 1, 2, \dots \\ X_n(x) \in \left\{1, \cos\left(\dfrac{n\pi}{L} x\right), \sin\left(\dfrac{n\pi}{L} x\right)\right\} \end{cases}$$

Mixed Boundary Value Problem

$$\left.\begin{matrix} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X'(L) \end{matrix}\right\} \Rightarrow \begin{cases} \lambda_k = \dfrac{(2k+1)\pi}{2L}, & k = 1, 2, \dots \\ X_n(x) = \sin\left(\dfrac{(2k+1)\pi x}{2L}\right) \end{cases}$$

and

$$\left.\begin{matrix} X'' + \lambda^2 X = 0 \\ X'(0) = 0 = X(L) \end{matrix}\right\} \Rightarrow \begin{cases} \lambda_k = \dfrac{(2k+1)\pi}{2L}, & k = 1, 2, \dots \\ X_n(x) = \cos\left(\dfrac{(2k+1)\pi x}{2L}\right) \end{cases}$$

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