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UNIFORMLY ASYMPTOTIC SOLUTIONS OF DIFFERENCE EQUATION

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ABSTRACT

In this paper, we attempt to obtain criteria for stability of the trivial solution of the first order difference equation applying various conditions in terms of Lyapunov function.

INTRODUCTION

In the recent years the theory and applications of difference equations are found to be more useful in the engineering field. Agarwal [1], Kelley and Peterson [2] developed the theory of difference equations and difference inequalities. Some differential and integral inequalities are given in [3]. K. L. Bondar contributed some difference inequalities, solutions of summation equations and some summation inequalities in [4, 5, 6]. Some comparison results in difference equations are given by A. B. Jadhav, P. U. Chopade and K. L. Bondar in [7]. Some stability criterion of solutions for the first order difference equation applying various conditions is given by P. U. Chopade in [8]. In this paper, we attempt to obtain criteria for stability of the trivial solution applying various conditions in terms of Lyapunov function of the first order difference equation

$$\Delta x(t) = f(t,\ x), \qquad x(t_0) = x_0, \quad t_0 \geq 0,$$

where $f \in C[J \times S_\rho, R_+]$, $J = \{t_0, t_0+1, t_0+2,\ \ldots, t_0+a\}$, $t_0 \in R_+$, the set of all nonnegative real numbers, S_ρ being the set

$$S_\rho = \{x \in R, |x| < \rho\}.$$

Definitions and Preliminary Notes

Let $x(t, t_0, x_0)$ be any solution of the difference equation

$$\Delta x(t) = f(t,x), \quad x(t_0) = x_0, \quad t_0 \geq 0. \qquad (2.1)$$

Assume that $f(t,0) = 0, t \in J$, so that $x = 0$ is a trivial solution of (2.1) through $(t_0, 0)$. We list a few definitions concerning the stability of the trivial solution.

Definition 2.1 For $V \in C[J \times R, R_+]$, we define the function

$$\Delta^+ V(t,x) = \sup_{t\in J}[V(t+1, x + f(t,x)) - V(t,x)] \qquad (2.2)$$

for $(t,x) \in J \times R$.

Definition 2.2 The trivial solution $x = 0$ of (2.1) is

(S_1) equistable if, for each $\epsilon > 0$, $t_0 \in J$, there exists a positive function $\delta = \delta(t_0,\ \epsilon)$ that is continuous in t_0 for each ϵ such that the inequality

$$|x_0| \leq \delta$$

implies

$$|\,x(t,t_0,x_0)| < \epsilon, \quad t \geq t_0;$$

(S_2) uniformly stable if the δ in (S_1) is independent of t_0;

(S_3) quasi-equi asymptotically stable if, for each $\epsilon > 0$, $t_0 \in J$, there exist positive numbers $\delta_0 = \delta_0(t_0)$ and $T = T(t_0,\ \epsilon)$ such that, for $t \geq t_0 + T$ and $|x_0| \leq \delta_0$,

$$|\,x(t,t_0,x_0)| < \epsilon;$$

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(S_4) quasi uniformly asymptotically stable if the numbers δ_0 and T in (S_3) are independent of t_0;

(S_5) equi-asymptotically stable if (S_1) and (S_3) hold simultaneously;

(S_6) uniformly asymptotically stable if (S_2) and (S_4) hold together.

It is convenient to introduce certain classes of monotone functions.

Definition 2.3 A function $\varphi(r)$ is said to belong to the class K if $\varphi \in C\,[[0,\rho), R_+]$, $\varphi(0) = 0$, and $\varphi(r)$ is strictly monotone increasing in r.

Definition 2.4 A function $V(t,x)$ with $V(t,0) \equiv 0$ is said to be positive definite if there exists a function $\varphi(r) \in K$ such that the relation

$$V(t,x) \geq \varphi(|x|)$$

is satisfied for $(t,\ x) \in J \times S_\rho$.

Definition 2.5 A function $V(t,x) \geq 0$ is said to be decrescent if a function $\varphi(r) \in K$ exists such that

$$V(t,x) \leq \varphi(|x|), (t,\ x) \in J \times S_\rho.$$

Definition 2.6 A function $V \in C[J \times S_\rho, R_+]$ is said to be locally Lipschitzian in x, if for each $(t,x) \in J \times S_\rho$ there exists a constant $M > 0$ and $\delta_0 > 0$ such that $|x - x_0| < \delta_0$, implies

$$|V(t,x) - V(t,x_0)| \leq M|x - x_0|.$$

Definition 2.7 Let $r(t)$ be any solution of (2.1) on J. Then $r(t)$ is said to be maximal solution of (2.1), if every solution $x(t)$ of (2.1) existing on J, the inequality $x(t) \leq r(t)$ holds for $t \in J$.

Definition 2.8 The function $V(t,x)$ is said to be mildly unbounded if, for every $T > 0, V(t,x) \to \infty$ as $|x| \to \infty$ uniformly for $t \in [0,T]$.

Definition 2.9 The function $g(t,u)$ is said to possess a mixed quasi-monotone property if the following conditions hold:

(i) $g_p(t,u)$ is nondecreasing in $u_j, j = 1,2,\dots,k, j \neq p$, and nonincreasing in u_q.

(ii) $g_q(t,u)$ is nonincreasing in u_p, and nondecreasing in $u_j,\ \ j = k+1, k+2, \dots, n,\ \ j \neq q$.

Evidently, the particular cases $k = n$ and $k = 0$ in the mixed quasi-monotone property correspond to quasi-monotone nondecreasing and quasi-monotone nonincreasing properties of the function $g(t,u)$ respectively. Furthermore, $g(t,u)$ is said to possess mixed monotone property if, in conditions (i) and (ii), $j \neq p, j \neq q$ are not demanded.

Theorem 2.1 [3] Let $g \in [E, R^n]$, where E is an open (t,u)-set in R^{n+1}. Suppose that g is a quasi-monotone nondecreasing in u. Let $[t_0,\ t_0 + a)$ be the largest interval of existence of the maximal solution $r(t)$ of

$$\Delta u(t) = g(t,\ u), \qquad u(t_0) = u_0.$$

Let

$m \in C[[t_0,\ t_0 + a),\ R^n], (t, w(t)) \in E, t \in [t_0,\ t_0 + a)$, and for a fixed derivative, the inequality

$$\Delta m(t) \leq g(t,\ m(t)) \tag{2.3}$$

holds for $t \in [t_0,\ t_0 + a)$. Then

$$m(t_0) \leq u_0 \tag{2.4}$$

implies

$$m(t) \leq r(t),\ t \in [t_0,\ t_0 + a). \tag{2.5}$$

Remark: If, in Theorem 2.1, the inequalities (2.3) and (2.4) are reversed, then the conclusion (2.5) is to be replaced by

$$m(t) \geq y(t),\ t \in [t_0,\ t_0 + a),$$

where $y(t)$ is the minimum solution of (2.1)

Main Comparison Theorem

The following theorem plays an important role whenever we use Lyapunov functions.

Theorem 3.1 Let $V \in C[J \times S_\rho, R_+]$ and $V(t,x)$ be locally Lipschitzian in x. Assume that the function $\Delta^+ V(x,t)$ defined by (2.2) satisfies the inequality

$$\Delta^+ V(t,x) \leq g(t, V(t,x)),\ (t,x) \in J \times S_\rho, \tag{3.1}$$

where $g \in C[J \times R_+,\ R\ ]$, and the function $g(t,u)$ is quasi-monotone nondecreasing in u, for each fixed $t \in J$. Let $r(t,t_0,u_0)$ be the maximal solution of the difference equation

$$\Delta u = g(t,u), u(t_0) = u_0 \geq 0,\ t_0 \geq 0, \tag{3.2}$$

existing to the right of t_0. If $x(t) = x(t,t_0,x_0)$ is any solution of (2.1) such that

$$V(t_0,x_0) \leq u_0, \tag{3.3}$$

then, as far as $x(t)$ exists to the right of t_0, we have

$$V(t, x(t,t_0,x_0)) \leq r(t,t_0,u_0) \tag{3.4}$$

Proof: Let $x(t,t_0,x_0)$ be any solution of (2.1) such that $V(t_0,x_0) \leq u_0$. Define the function $m(t)$ by

$$m(t) = V(t, x(t,t_0,x_0)).$$

Then, using the hypothesis that $V(t,x)$ satisfies Lipschitz's condition in x, we obtain, the inequality

$$m(t+1) - m(t) \leq K\left|x(t+1) - x(t) - f(t,x(t))\right| + V\left(t+1, x(t) + f(t,x(t))\right) - V(t,x(t)),$$

where K is the local Lipschitz constant. This, together with (2.1) and (3.1), implies the inequality

$$\Delta^+ m(t) \leq g(t, m(t)).$$

Moreover, $m(t_0) \leq u_0$. Hence by Theorem 2.1, we have

$$m(t) \leq r(t,t_0,u_0)$$

as far as $x(t)$ exists to the right of t_0, proving the desired relation (3.4).

We can now state a global existence theorem.

Theorem 3.2 Assume that $V \in C[J \times R, R_+], V(t,x)$ is locally Lipschitzian in x and $\sum_{i=1}^{N} V_i(t,x)$ is mildly unbounded. Suppose that $g \in C[J \times R, R_+], g(t,u)$ is quasi-monotonic nondecreasing in u for each fixed $t \in J$, and $r(t,t_0,u_0)$ is the maximal solution of (3.2) existing for $t \geq t_0$. If $f \in C[J \times R, R]$ and

$$\Delta^+ V(t,x) \leq g(t, V(t,x)), (t,x) \in J \times R,$$

then every solution

$$x(t) = x(t, t_0, x_0)$$

of (2.1) exists in the future and (3.3) implies (3.4) for all $t \geq t_0$.

On the basis of Theorem 2.1 and the remark that follows, we can prove the following:

Theorem 3.3 Let $V \in C[J \times S_\rho, R_+]$ and $V(t, x)$ be locally Lipschitizian in *x*. Suppose that $g_1, g_2 \in C[J \times R_+, R], g_1(t, u), g_2(t, u)$ possess quasi-monotone nondecreasing property in *u* for each $t \in J$, and, for

$(t, x) \in J \times S_\rho$,

$$g_1\left(t, V(t, x)\right) \leq \Delta^+ V(t, x) \leq g_2\left(t, V(t, x)\right).$$

Let $r(t, t_0, u_0), \; \rho(t, t_0, v_0)$ be the maximal, minimal solutions of

$$\Delta u = g_2(t, u), \qquad u(t_0) = u_0,$$

$$\Delta v = g_1(t, u), \quad v(t_0) = v_0,$$

respectively such that

$$v_0 \leq V(t_0, x_0) \leq u_0.$$

Then, as far as

$$x(t) = x(t, t_0, x_0)$$

exists to the right of t_0, we have

$$\rho(t, t_0, v_0) \leq V(t, x(t)) \leq r(t, t_0, u_0),$$

where $x(t)$ is any solution of (2.1).

Asymptotic Stability

An approach that is extremely fruitful in proving asymptotic stability is to modify Lyapunov's original theorem without demanding $\Delta^+ V(t, x)$ to be negative definite. The theorem that follows takes care of the general case of $f(t, x)$ and requires two Lyapunov functions.

Theorem 4.1 Suppose that the following conditions hold:

(i) $f \in C\left[J \times S_\rho, R_+\right], f(t, 0) = 0$, and $f(t, x)$ is bounded on $J \times S_\rho$.

(ii) $V_1 \in C\left[J \times S_\rho, R_+\right]$, $V_1(t, x)$ is positive definite, decrescent, locally Lipschitzian in *x*, and

$$\Delta^+ V_1(t, x) \leq w(x) \leq 0, \;\; (t, x) \in J \times S_\rho,$$

where $w(x)$ is continuous for $x \in S_\rho$.

(iii) $V_2 \in C\left[J \times S_\rho, R_+\right]$ and $V_2(t, x)$ is bounded on $J \times S_\rho$ and is locally Lipschitzian in *x*. Furthermore, given any number, α, $0 < \alpha < \rho$, there exist positive numbers

$$\xi = \xi(\alpha) > 0, \eta = \eta(\alpha) > 0, \eta < \alpha,$$

such that

$$\Delta^+ V_2(t, x) > \xi, \text{ for } \alpha < |x| < \rho \text{ and } d(x, E) < \eta, t \geq 0,$$

where $E = [x \in S_\rho : w(x) = 0]$ and $d(x, E)$ is the distance between the point *x* and the set *E*. Then, the trivial solution of (2.1) is uniformly asymptotically stable.

Proof: Let $\epsilon > 0$ and $t_0 \in J$ be given. Since $V_1(t, x)$ is positive definite and decrescent, there exists functions $a, b \in K$ such that

$$b(|x|) \leq V_1(t, x) \leq a(|x|), \;\; (t, x) \in J \times S_\rho. \qquad (4.1)$$

We choose $\delta = \delta(\epsilon)$ so that

$$b(\epsilon) > a(\delta). \qquad (4.2)$$

Then, we can conclude that the trivial solution of (2.1) is uniformly stable.

Let us now fix $\epsilon = \rho$ and define $\delta_0 = \delta(\rho)$. Let $0 < \epsilon < \rho, t_0 \in J$, and define $\delta = \delta(\epsilon)$ be the same δ obtained in (4.2) for uniform stability. Assume that $|x| < \delta_0$. To prove uniform asymptotic stability of the solution $x = 0$, it is enough to show that there exists a $T = T(\epsilon)$ such that, for some $t^* \in [t_0, t_0 + T]$, we have

$$|x(t^*, t_0, x_0)| < \delta.$$

This we achieve in a number of stages:

(a) If $d[x(t_1), x(t_2)] > r > 0, t_2 > t_1$, then

$$r \leq M n^{\frac{1}{2}}(t_2 - t_1), \qquad (4.3)$$

where

$$|f(t, x)| \leq M, (t, x) \in J \times S_\rho.$$

For, consider

$$|x_i(t_1) - x_i(t_2)| \leq \sum_{t_1}^{t_2 - 1} |\Delta x_i(s)|$$

$\leq \sum_{t_1}^{t_2-1} |f_i(s, x(s))|, (i = 1, 2, \ldots, n),$
$\leq M(t_2 - t_1),$
and therefore

$$r < d[x(t_1), x(t_2)] = \{[x_1(t_1) - x_1(t_2)]^2 + [x_2(t_1) - x_2(t_2)]^2 + \cdots + [x_n(t_1) - x_n(t_2)]^2\}^{\frac{1}{2}} \leq M n^{\frac{1}{2}}(t_2 - t_1).$$

(b) By assumption (iii), given $\delta = \delta(\epsilon), 0 < \epsilon < \rho$, there exist $\xi = \xi(\epsilon), \eta = \eta(\epsilon), \eta < \delta$ such that

$$\Delta^+ V(t, x) > \xi, \;\; \delta < |x| < \rho, \;\; d(x, E) < \eta, \;\; t \geq 0.$$

Let us consider the set

$$U = \left[x \in S_\rho : \delta < |x| < \rho, d(x, E) < \eta\right],$$

and let

$$sup_{|x|<\rho, t\geq 0} V_2(t, x) = L.$$

Assume that, at $t = t_1, \; x(t_1) = x(t_1, t_0, x_0) \in U$. Then, for $t > t_1$, we have, letting
$m(t) = V_2(t, x(t)),$

$$\Delta^+ V_2(t, x(t)) > \xi,$$

because of condition (iii) and the fact that $V_2(t, x)$ satisfies a Lipschitz condition in x locally. Thus,

$$m(t) - m(t_1) = \sum_{t_1}^{t-1} \Delta^+ m(s),$$

and hence

$$m(t) + m(t_1) \geq \sum_{t_1}^{t-1} \Delta^+ m(s) \geq \sum_{t_1}^{t-1} \Delta^+ V_2(s, x(s)) > \xi(t - t_1)$$

as long as $x(t)$ remains in *U*. This inequality can simultaneously be realized with $m(t) \leq L$ only if $t < t_1 + \frac{2L}{\xi}$. It therefore follows that there exists a $t_2, t_1 < t_2 \leq t_1 + \frac{2L}{\xi}$ such that $x(t_2)$ is on the boundary of the set *U*. In other words, $x(t)$ cannot stay permanently in the set *U*.

(c) Consider the sequence $\{t_k\}$ such that

$$t_k = t_0 + k\frac{2L}{\xi}, (k = 0, 1, 2, \dots).$$

Set $n(t) = V_1(t, x(t))$. Then, by assumption (ii), we have

$$\Delta^+ n(t) \leq \Delta^+ V_1(t, x(t)) \leq 0.$$

We let

$$\lambda = \inf[|w(x)|,\ \delta < |x| < \rho,\ d(x, E) \geq \frac{\eta}{2}],$$

and $\lambda_1 = \frac{\lambda_\eta}{2Mn^{\frac{1}{2}}}$. Suppose that $x(t)$ is such that, for $t_k \leq t \leq t_{k+2}$, $\delta < |x| < \rho$. If for $t_k \leq t \leq t_{k+1}$, we have $\delta < |x| < \rho$ and $d(x, E) \geq \frac{\eta}{2}$, then, using assumption (ii) together with the definition of the set *E*, we obtain

$$\begin{aligned} n(t_{k+2}) - n(t_k) &= \sum_{t_k}^{t_{k+2}-1} \Delta^+ n(s) \\ &\leq \sum_{t_k}^{t_{k+2}-1} \Delta^+ V_1(s, x(s)) \\ &\leq \sum_{t_k}^{t_{k+1}} \Delta^+ V_1(s, x(s)) \\ &\quad + \sum_{t_{k+1}+1}^{t_{k+2}-1} \Delta^+ V_1(s, x(s)) \\ &\leq -\lambda(t_{k+1} - t_k) \\ &= -\lambda\frac{2L}{\xi}. \end{aligned} \qquad (4.4)$$

On the other hand, if it happens that, for $t_k \leq t_1 \leq t_{k+1}$,

$$\delta < |x\ (t_1)| < \rho, \quad d(x(t_1), E) \geq \frac{\eta}{2},$$

then there exists a t_3, $t_1 < t_3 \leq t_1 + \frac{2L}{\xi}$ such that $d[x(t_3), E] = \eta$, in the view of (b). It follows that there also exists a t_4, $t_1 \leq t_4 < t_3$ satisfying

$$d(x(t_4), E) = \frac{\eta}{2}.$$

These considerations lead to $d(x(t_3), x(t_4)) \geq \frac{\eta}{2}$, and hence we obtain, because of (a),

$$\frac{\eta}{2} \leq Mn^{\frac{1}{2}}(t_3 - t_4),$$

which implies

$$\frac{\eta}{2Mn^{\frac{1}{2}}} \leq (t_3 - t_4) \leq \frac{2L}{\xi}. \qquad (4.5)$$

Moreover,

$$\begin{aligned} n(t_3) - n(t_1) &\leq \sum_{t_1}^{t_4-1} \Delta^+ V_1(s, x(s)) \\ &\quad + \sum_{t_4}^{t_3-1} \Delta^+ V_1(s, x(s)) \\ &\leq -\lambda(t_3 - t_4) \\ &= \frac{-\lambda_\eta}{2Mn^{\frac{1}{2}}} \\ &= -\lambda_1. \end{aligned}$$

Since $n(t)$ is nonincreasing function, we have

$$n(t_{k+2}) \leq n(t_3) \leq n(t_1) - \lambda_1 \leq n(t_k) - \lambda_1.$$

Also, on the basis of (4.5), we obtain from (4.4) that

$$n(t_{k+2}) \leq n(t_k) - \lambda_1.$$

Thus, in any case

$$V_1(t_{k+2},\ x(t_{k+2})) \leq V_1(t_k,\ x(t_k)) - \lambda_1.$$

Choose an integer k^* such that $\lambda_1 k^* > a(\delta_0)$ and $T = T(\epsilon) = 4k^* \frac{L}{\zeta(\epsilon)}$. Assume that, for

$$t_0 \leq t \leq t_0 + T,\ |x(t, t_0, x_0)| \geq \delta.$$

It then results from the preceding considerations that

$$\begin{aligned} V_1(t_0 + T, x(t_0 + T)) &\leq V_1(t_0, x_0) - \lambda_1 k^* \\ &\leq a(\delta_0) - \lambda_1 k^* \\ &\leq 0, \end{aligned}$$

which is incompatible with the positive definiteness of $V_1(t, x)$. Thus, there exists a $t^* \in [t_0, t_0 + T]$ satisfying

$$|x(t^*, t_0, x_0)| < \delta$$

and the proof is complete.

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