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**Research Article**

# THEORY OF RELATIVITY AND QUANTUM MECHANICS AS COMPLEMENTARY PARTS OF A UNITARY THEORY

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**ABSTRACT**

The starting point of this research is a contradiction of the conventional Schrödinger equation with one of the fundamental Hamilton equations – a minus sign, essential for the energy conservation, is missing. The full agreement of the Schrödinger equation with the Hamilton equations is obtained only when the Hamiltonian in the time dependent phases of the two wave packets representing a quantum particle in the coordinate and momentum spaces is replaced by its Lagrangian. We consider the Universe as a distribution of 'intrinsic' matter, characterized by curvilinear time-space coordinates, curved in a system of other coordinates, by an 'extrinsic' matter, with a density as another matter coordinate. According to the general theory of relativity, any acceleration of an extrinsic matter differential element in an 'extrinsic' (non-gravitational) field is perpendicular to the velocity. This characteristic describes a matter propagation in planes perpendicular to the velocity. This dynamics can be described by two Fourier conjugated wave packets, with a condition of quantization asserting that the space integral of the matter density is equal to the rest mass in the coefficient of the time dependent phases of these wave packets, which, according to their group velocities, appears as a Lagrangian. In this framework, fundamental physical problems are reconsidered by using the general theory of relativity in Dirac's formulation, for the description of the quantum dynamics. Although in this paper we develop an essentially relativistic theory, in the proper system of a quantum particle we consider only small velocities of its differential matter elements, otherwise this particle being shattered in space, as the notion of 'particle' has no more any sense. We show that the Schwarzschild solution with a singularity is only an approximate one, since the dynamics of a differential matter element is always joined to the dynamics of the other matter elements of a quantum particle, and of other quantum particles always present in the realistic cases. These matter elements perturb the gravitational field considered for the Schwarzschild solution, leading to a penetrability of the boundary of a black hole, from the outside for an absorption rate, and from the inside, for an evaporation rate. We consider black particles with phases including only relativistic Lagrangians depending on the rest masses, and 'visible' particles, including other interaction terms depending on 'charges'. We obtain the Lorentz force and the Maxwell equations as properties of a field interacting with a quantum particle, relativistic quantum equations with spin interaction, the spin of the extrinsic matter of a quantum particle, and the spin of the intrinsic component of this particle, we call 'graviton'.



## INTRODUCTION

In the conventional quantum mechanics [1]-[3], a quantum particle with an energy $E$ is generally described by a Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi_E\left(\vec{r},t\right)=H_0\left(\vec{p},\vec{r}\right)\psi_E\left(\vec{r},t\right)=E\psi_E\left(\vec{r},t\right), \qquad (1)$$

depending on a Hamiltonianequal to this energy,

$$H_0\left(\vec{p},\vec{r}\right)=T\left(\vec{p}\right)+U\left(\vec{r}\right)=E\,, \qquad (2)$$

as a sum of the kinetic energy $T\left(\vec{p}\right)$ with the potential energy $U\left(\vec{r}\right)$.The general solution of this equation is a wave packet of the form

$$\psi_E\left(\vec{r},t\right)=\frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\varphi_0\left(\vec{p},t\right)e^{\frac{\mathrm{i}}{\hbar}\left\{\vec{p}\vec{r}-\left[T\left(\vec{p}\right)+U\left(\vec{r}\right)\right]t\right\}}\mathrm{d}^3\vec{p}\,, \qquad (3)$$

with the group velocity according to the first Hamilton equation

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$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{r} = \frac{\partial}{\partial \vec{p}} T\left(\vec{p}\right) = \frac{\partial H_0}{\partial \vec{p}} . \tag{4}$$

However, we notice that the Fourier conjugate wave packet in the momentum space,

$$\varphi_0\left(\vec{p},t\right) = \frac{1}{\left(2\pi\hbar\right)^{3/2}} \int \psi_E\left(\vec{r},t\right) e^{-\frac{\mathrm{i}}{\hbar}\left\{\vec{p}\vec{r} - \left[T\left(\vec{p}\right)+U\left(\vec{r}\right)\right]t\right\}} \mathrm{d}^3\vec{r} , \tag{5}$$

has a group velocity,

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{p} = \frac{\partial H_0}{\partial \vec{r}} = \underline{\frac{\partial}{\partial \vec{r}} U\left(\vec{r}\right)}, \tag{6}$$

contradictory to the second Hamilton equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{p} = -\frac{\partial H_0}{\partial \vec{r}} = \underline{-\frac{\partial}{\partial \vec{r}} U\left(\vec{r}\right)} . \tag{7}$$

A minus sign is missing, as this minus is essential for the energy conservation,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial H_0}{\partial \vec{r}}\frac{\mathrm{d}}{\mathrm{d}t}\vec{r} + \frac{\partial H_0}{\partial \vec{p}}\frac{\mathrm{d}}{\mathrm{d}t}\vec{p} = 0 . \tag{8}$$

We conclude that for a correct description of the quantum dynamics, in the wave packets (3) and (5) the Hamiltonian must be replaced by the Lagrangian [4]-[6]:

$$H_0\left(\vec{p},\vec{r}\right) = T\left(\vec{p}\right) + U\left(\vec{r}\right) \to L_0\left(\vec{r},\dot{\vec{r}}\right) = T\left(\vec{p}\right) - U\left(\vec{r}\right) \cdot \tag{9}$$

We obtain wave functions in full agreement with the Hamilton equations (4) and (7),

$$\begin{aligned}
\psi_0\left(\vec{r},t\right) &= \frac{1}{\left(2\pi\hbar\right)^{3/2}} \int \varphi_0\left(\vec{p},t\right) e^{\frac{\mathrm{i}}{\hbar}\left\{\vec{p}\vec{r} - \left[T\left(\vec{p}\right)-U\left(\vec{r}\right)\right]t\right\}} \mathrm{d}^3\vec{p} = \frac{1}{\left(2\pi\hbar\right)^{3/2}} \int \varphi_0\left(\vec{p},t\right) e^{\frac{\mathrm{i}}{\hbar}\left[\vec{p}\vec{r} - L_0\left(\vec{r},\dot{\vec{r}}\right)t\right]} \mathrm{d}^3\vec{p} \\
\varphi_0\left(\vec{p},t\right) &= \frac{1}{\left(2\pi\hbar\right)^{3/2}} \int \psi_0\left(\vec{r},t\right) e^{-\frac{\mathrm{i}}{\hbar}\left\{\vec{p}\vec{r} - \left[T\left(\vec{p}\right)-U\left(\vec{r}\right)\right]t\right\}} \mathrm{d}^3\vec{r} = \frac{1}{\left(2\pi\hbar\right)^{3/2}} \int \psi_0\left(\vec{r},t\right) e^{-\frac{\mathrm{i}}{\hbar}\left\{\vec{p}\vec{r} - L_0\left(\vec{r},\dot{\vec{r}}\right)t\right\}} \mathrm{d}^3\vec{r} .
\end{aligned} \tag{10}$$

These equations suggest a relativistic description of the quantum dynamics, by wave functions with time dependent phase elements of the form [5]-[6],

$$L_0\left(\dot{\vec{r}}\right)\mathrm{d}t = -M_0 c^2 \sqrt{1 - \frac{\dot{\vec{r}}^2}{c^2}}\mathrm{d}t = -M_0 c \mathrm{d}s . \tag{11}$$

Based on this equation, we reformulated the relativistic principle of the invariance of the time-space interval $\mathrm{d}s$ as a relativistic quantum principle of invariance of the time dependent phase of the wave function of a quantum particle [6]-[9]. From the dynamical equations as group velocities of the wave functions in the coordinate and the momentum spaces, we obtained the Maxwell equations, the spin interaction, and Dirac's relativistic equations with rest mass, of a quantum particle in electromagnetic field.

In this paper we consider the more general case of a Universe as a distribution of 'intrinsic' matter characterized by time-space coordinates, and curved in a system of other coordinates by'extrinsic' matter distributions, with other coordinates, of mass and other possible charges. We believe that such a representation enables a better understanding of the open quantum physics [10]-[14], as a unitary dynamics of a matter-field system which, in this way, takes the form of a continuous distribution of matter – some matter excitations, as the photons or the phonons, can be hardly understood as 'quantum particles' with coordinate probabilities described by 'wave functions'. In section 2, according to the general theory of relativity [15], we find that any acceleration of a matter element in a non-gravitational field is perpendicular to the velocity. We consider a positively defined density of matter, proportional to the square of a function of distribution. Taking into account a Fourier series expansion of this function, called 'wave function', we define the quantization condition as an equality of the total mass, as the space integral of the density, with the rest mass in the Lagrangian of the time dependent phase of this wave function. In section 3, we consider a quantum particle with an electric charge, interacting with an electromagnetic field described by a vector potential conjugated to the coordinates, and an electric potential conjugated to time. From the terms of the electromagnetic interaction in the time dependent phase of a quantum particle, we obtain the Lorentz force as a function of the electric and magnetic fields, defined as functions of the two electromagnetic potentials, and the Maxwell equations for these fields. In section 4, we consider the Lagrangian as a function of the momentum-velocity product and Hamiltonian, and obtain relativistic equations including the rest mass and the particle momentum and velocity. For an energy eigenvalue, we derive an online are equation with as pin interaction which, for a non-relativistic velocity, takes the linear form of the conventional Schrödinger equation with the spin interaction. In section 5, we derive the Schwarzschild solution for the metric tensor in a constant central gravitational field, and show that, due to the variations of this field induced by the other matter elements of a quantum particle and of other particles always present in the realistic cases, the boundary of a black hole can be passed in the both directions. In section 6, we obtain dynamic equations for a quantum particle in a gravitational wave. For the time-space component of the extrinsic matter of a quantum particle, called graviton, we obtain a proper rotation with a spin 2, as the density dynamics of the matter in the time-space coordinates of this particle includes a proper rotation with a half-integer spin for an anti-symmetric wave function (Fermion), or an integer spin for a symmetric wave function (Boson), according to the spin-statistics relation. In section 7, we give a summary.

### *Quantum Particle as a Distribution of Matter*

In a time-space reference system $S$ with the coordinates

$$x = \left(x^\alpha\right) = \left(x^0 = ct, x^1, x^2, x^3\right) = \left(x^0 = ct, x^i\right), \tag{12}$$

we consider a distribution of extrinsic matter with a positively defined density by a distribution function $\psi\left(x^i,t\right)$,

$$\rho\left(x^i,t\right) = M\left|\psi\left(x^i,t\right)\right|^2, \tag{13}$$

with the normalization condition

$$\int\left|\psi\left(x^i,t\right)\right|^2 \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = 1, \tag{14}$$

And the total mass $M$ ,

$$\int \rho\left(x^i,t\right)\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = M\int\left|\psi\left(x^i,t\right)\right|^2 \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = M \,. \qquad (15)$$

In the case of the special relativity, of an inertial system $S$, and of another time-space reference system $S'$ with parallel axes,

$$x' = \left(x^{\alpha'}\right) = \left(x^{0'} = ct', x^{1'}, x^{2'}, x^{3'}\right) = \left(x^{0'} = ct', x^{i'}\right), \qquad (16)$$

moving with a constant velocity $V$ in the direction $x^1 \parallel x^{1'}$ (**Fig. 1**),

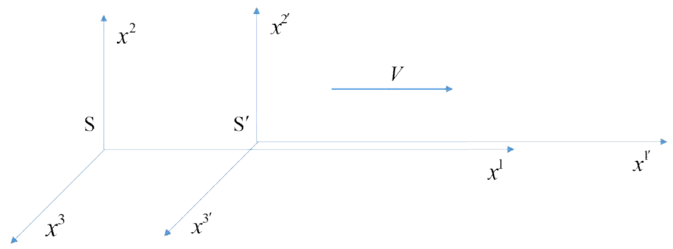


**Fig 1A** system of coordinates , in a system of coordinates with parallel axes, moving with a velocity in the - direction.

the coordinate transformation has a simpler form:

$$\begin{cases} x^0 = x^0\left(x^{0'}, x^{1'}\right) \\ x^1 = x^1\left(x^{0'}, x^{1'}\right) \\ x^2 = x^{2'} \\ x^3 = x^{3'}, \end{cases}$$

(17)

with the relativistic condition of the invariance of any time-space interval,

$$\mathrm{d}s^2 = \mathrm{d}x^{0^2} - \mathrm{d}x^{1^2} - \mathrm{d}x^{2^2} - \mathrm{d}x^{3^2} = \mathrm{d}s'^2 = \mathrm{d}x^{0'^2} - \mathrm{d}x^{1'^2} - \mathrm{d}x^{2'^2} - \mathrm{d}x^{3'^2}\,. \qquad (18)$$

From these expressions, we obtain the coordinate transformation

$$\begin{cases} \mathrm{d}x^0 = \sqrt{1 + x^1{}_{,0'}{}^2}\,\mathrm{d}x^{0'} + x^1{}_{,0'}\mathrm{d}x^{1'} \\ \mathrm{d}x^1 = x^1{}_{,0'}\mathrm{d}x^{0'} + \sqrt{1 + x^1{}_{,0'}{}^2}\,\mathrm{d}x^{1'} \end{cases} \qquad (19)$$

with the determinant $1$, and the reverse transformation

$$\begin{cases} \mathrm{d}x^{0'} = \sqrt{1 + x^1{}_{,0'}{}^2}\,\mathrm{d}x^0 - x^1{}_{,0'}\mathrm{d}x^1 \\ \mathrm{d}x^{1'} = -x^1{}_{,0'}\mathrm{d}x^0 + \sqrt{1 + x^1{}_{,0'}{}^2}\,\mathrm{d}x^1. \end{cases} \qquad (20)$$

We notice that from the second expression (20) we obtain the physical value $V$ as a function of the coefficients of this transformation:

$$\frac{V}{c} = \left.\frac{\mathrm{d}x^1}{\mathrm{d}x^0}\right|_{\mathrm{d}x^{1'}=0} = \frac{x^1{}_{,0'}}{\sqrt{1 + x^1{}_{,0'}{}^2}}\,. \qquad (21)$$

With the expression

$$1 - \frac{V^2}{c^2} = \frac{1}{1 + x^1{}_{,0'}{}^2}\,, \qquad (22)$$

we obtain the transformation coefficient

$$\sqrt{1 + x^1{}_{,0'}{}^2} = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\,. \qquad (23)$$

With this coefficient and the expression (21) for the other coefficient of the transformation (19), for the intrinsic coordinates of a distribution of matter (13), we obtain the Lorentz transformation

$$\begin{cases} \mathrm{d}t = \dfrac{\mathrm{d}t' + \dfrac{V}{c^2}\mathrm{d}x^{1'}}{\sqrt{1 - \dfrac{V^2}{c^2}}} \\ \mathrm{d}x^1 = \dfrac{\mathrm{d}x^{1'} + V\mathrm{d}t'}{\sqrt{1 - \dfrac{V^2}{c^2}}}, \qquad \mathrm{d}x^2 = \mathrm{d}x^{2'}, \qquad \mathrm{d}x^3 = \mathrm{d}x^{3'}. \end{cases} \qquad (24)$$

In the case of the general relativity, the matter dynamics is described by a time-space interval $\mathrm{d}s$ of the form

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \mathrm{d}x_\nu\mathrm{d}x^\nu \quad / \quad \mathrm{d}s^2\,. \qquad (25)$$

By dividing the square of this interval by itself, we obtain the fundamental relation of the relativistic dynamics of a differential matter element,

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 1\,, \qquad (26)$$

for its velocity in its proper time,

$$\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\,. \qquad (27)$$

Since any covariant derivative of the metric tensor is null [15],

$$g_{\mu\nu:\sigma} = 0\,, \qquad (28)$$

from the covariant derivative of the fundamental relation (26) we obtain the relation

$$\dot{x}^\mu_{:\sigma}\dot{x}_\mu = 0\,,$$

(29)

which means that any covariant derivative of the velocity is perpendicular to this velocity. On the other hand, we notice that any acceleration $A^\mu$ induced by an external (non-gravitational) field, arises as an additional term to the geodesic equation of the inertial-gravitational dynamics,

$$\dot{x}^\mu_{,\nu}\dot{x}^\nu = -\Gamma^\mu_{\nu\sigma}\dot{x}^\nu\dot{x}^\sigma + A^\mu\,. \qquad (30)$$

In this way, we find that as in a gravitational field a differential matter element gets only an ordinary acceleration, the covariant acceleration being null, in an external field it gets also a covariant acceleration:

$$A^{\mu} = \left(\dot{x}^{\mu}_{,\nu} + \Gamma^{\mu}_{\nu\sigma}\dot{x}^{\sigma}\right)\dot{x}^{\nu} = \dot{x}^{\mu}_{;\nu}\dot{x}^{\nu}. \tag{31}$$

From the scalar product of this expression with the velocity, and the general equation (29), we find that the acceleration induced by an external field is perpendicular to the velocity,

$$A^{\mu}\dot{x}_{\mu} = \underbrace{\dot{x}^{\mu}_{;\nu}\dot{x}_{\mu}}_{0}\dot{x}^{\nu} = 0. \tag{32}$$

This means that matter moves in planes perpendicular to the velocity, as in **Fig. 2** of a particle in a central field.

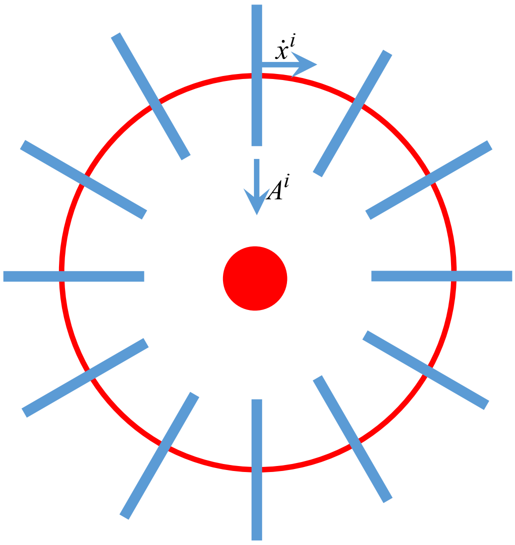


**Fig 2** Matter distribution in a central field, accelerating this matter in planes perpendicular to the velocity.

In this case, that distribution function $\psi\left(x^i,t\right)$ of the density (13),

$$\rho\left(x^i,t\right) = \underline{\underline{M}}\left|\psi\left(x^i,t\right)\right|^2, \tag{33}$$

with the normalization condition (14), can be considered as a Fourier series expansion in space,

$$\psi\left(x^i,t\right) = \frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\varphi\left(P_j,t\right)e^{\frac{i}{\hbar}\left[P_jx^j - L_0\left(x^{\alpha},\dot{x}^{\alpha}\right)t\right]}\mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P_3, \tag{34}$$

with a relativistic Lagrangian describing the dynamics of a particle with the rest mass $M_0$,

$$L_0\left(x^{\alpha},\dot{x}^{\alpha}\right) = -\underline{\underline{M_0}}c^2\sqrt{g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}}, \tag{35}$$

and the momentum

$$P_i = \frac{\partial L_0\left(x^{\alpha},\dot{x}^{\alpha}\right)}{c\partial\dot{x}^i}. \tag{36}$$

From the reverse Fourier transform in the conjugated momentum space

$$\varphi\left(P_j,t\right) = \frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\psi\left(x^i,t\right)e^{-\frac{i}{\hbar}\left[P_jx^j - L_0\left(x^{\alpha},\dot{x}^{\alpha}\right)t\right]}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3, \tag{37}$$

we obtain the group velocity as a Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}P_i = c\frac{\mathrm{d}}{\mathrm{d}x^0}P_i = \frac{\partial L_0\left(x^{\alpha},\dot{x}^{\alpha}\right)}{\partial x^i}. \tag{38}$$

We notice that the Fourier series expansions (34) and (37), with the normalization condition (14) and the Lagrangian (35), describe the dynamics of the whole distribution of matter (33) only if

$$M = M_0, \tag{39}$$

we call quantization condition, while a distribution of extrinsic matter (33) satisfying this condition is called quantum particle.This means that a differential matter element of mass $\mathrm{d}M$ cannot exist isolated in Universe, but only as part of a quantum particle,which consists in a distribution of matter (33) described by wave functions of the forms (34) and (37), with the quantization condition (39).

### *Quantum Particle in Electromagnetic field*

In the previous section we mainly considered a black quantum particle characterized only by a rest mass $M_0$ interacting only with a gravitational field, as a time-space deformation, described by the metric tensor $g_{\mu\nu}$. Here we consider a quantum particle with an electric charge $e$ ,described by the wave functions

$$\begin{aligned}\psi\left(\vec{r},t\right) &= \frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\varphi\left(\vec{P},t\right)e^{\frac{i}{\hbar}\left[\vec{P}\vec{r} - L\left(\vec{r},\dot{\vec{r}},t\right)t\right]}\mathrm{d}^3\vec{P} \\ \varphi\left(\vec{P},t\right) &= \frac{1}{\left(2\pi\hbar\right)^{3/2}}\int\psi\left(\vec{r},t\right)e^{-\frac{i}{\hbar}\left[\vec{P}\vec{r} - L\left(\vec{r},\dot{\vec{r}},t\right)t\right]}\mathrm{d}^3\vec{r}.\end{aligned} \tag{40}$$

By this charge, the particle interacts with an electromagnetic field described by the vector potential $\vec{A}\left(\vec{r},t\right)$ conjugated to the spatial coordinates, which is in agreement with the Aharonov-Bohm effect [16], and the scalar potential $U\left(\vec{r}\right)$ conjugated to time,

$$L\left(\vec{r},\dot{\vec{r}},t\right)\mathrm{d}t = -M_0c^2\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}}\mathrm{d}t + e\vec{A}\left(\vec{r},t\right)\mathrm{d}\vec{r} - eU\left(\vec{r}\right)\mathrm{d}t. \tag{41}$$

For a quantum particle much smaller than the field non-uniformities, this expression can be linearized in the wave function phase,

$$L\left(\vec{r},\dot{\vec{r}},t\right) = -\mathrm{M_0c^2}\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}} + e\vec{A}\left(\vec{r},t\right)\dot{\vec{r}} - eU\left(\vec{r}\right). \tag{42}$$

For the canonical momentum

$$\vec{P} = \frac{\partial}{\partial\dot{\vec{r}}}L\left(\vec{r},\dot{\vec{r}},t\right) = \frac{M_0\dot{\vec{r}}}{\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}}} + e\vec{A}\left(\vec{r},t\right) = \underbrace{\vec{p}}_{\substack{\text{mechanical}\\ \text{momentum}}} + \underbrace{e\vec{A}\left(\vec{r},t\right)}_{\substack{\text{electromagnetic}\\ \text{momentum}}}, \tag{43}$$

composed of a mechanical momentum and an electromagnetic momentum, we obtain the Lagrange equation as the group velocity of the Fourier transform of the particle coordinate wave function (40),

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{P}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{\vec{r}}}L\left(\vec{r},\dot{\vec{r}},t\right)=\frac{\partial}{\partial\vec{r}}L\left(\vec{r},\dot{\vec{r}},t\right). \qquad (44)$$

With the Lagrangian (42) and the momentum (43) of a quantum particle in a time independent electric potential $U\left(\vec{r}\right)$, under the action of an electromagnetic radiation with the time dependent vector potential $\vec{A}\left(\vec{r},t\right)$,we obtain the Hamiltonian as a constant function we call the particle energy in electric potential,

$$\begin{aligned}H\left(\vec{P},\vec{r},t\right)&=\vec{P}\dot{\vec{r}}-L\left(\vec{r},\dot{\vec{r}},t\right)\\&=\frac{M_0\dot{\vec{r}}^2}{\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}}}+\underline{e\vec{A}\left(\vec{r},t\right)\dot{\vec{r}}}-\left(-M_0c^2\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}}+\underline{e\vec{A}\left(\vec{r},t\right)\dot{\vec{r}}}-eU\left(\vec{r}\right)\right)\\&=\frac{M_0c^2}{\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}}}+eU\left(\vec{r}\right)=E\left(\vec{r},\dot{\vec{r}}\right).\end{aligned} \qquad (45)$$

At the same time, in (43) we distinguish the mechanical momentum

$$\vec{p}=\frac{M_0\dot{\vec{r}}}{\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}}}=\vec{P}-e\vec{A}\left(\vec{r},t\right), \qquad (46)$$

as the first term of the Hamiltonian (45) takes the form of a function of this momentum,

$$\frac{M_0^2c^2}{1-\frac{\dot{\vec{r}}^2}{c^2}}=\frac{M_0^2\dot{\vec{r}}^2}{1-\frac{\dot{\vec{r}}^2}{c^2}}+M_0^2c^2=\vec{p}^2+M_0^2c^2. \qquad (47)$$

With these expressions, we obtain the Hamiltonian (45) as a function of the mechanical momentum $\vec{p}$, or of the canonical momentum $\vec{P}$, and the coordinates $\vec{r}$,

$$H\left(\vec{P},\vec{r},t\right)=c\sqrt{M_0^2c^2+\vec{p}^2}+eU\left(\vec{r}\right)=c\sqrt{M_0^2c^2+\left[\vec{P}-e\vec{A}\left(\vec{r},t\right)\right]^2}+eU\left(\vec{r}\right). \qquad (48)$$

At the same time, from (46) we obtain the mechanical force acting on a quantum particle with the wave functions (40) and the Lagrangian (42),

$$\vec{F}_e=\frac{\mathrm{d}}{\mathrm{d}t}\vec{p}=\frac{\mathrm{d}}{\mathrm{d}t}\vec{P}-e\frac{\mathrm{d}}{\mathrm{d}t}\vec{A}\left(\vec{r},t\right)=\frac{\mathrm{d}}{\mathrm{d}t}\vec{P}-e\frac{\partial}{\partial t}\vec{A}\left(\vec{r},t\right)-\underline{\underline{e\left(\dot{\vec{r}}\frac{\partial}{\partial\vec{r}}\right)\vec{A}\left(\vec{r},t\right)}}. \qquad (49)$$

From the group velocity of the second equation (40) with the Lagrangian (42), we obtain the first term of this force,

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{P}=\frac{\partial}{\partial\vec{r}}L\left(\vec{r},\dot{\vec{r}},t\right)=e\underline{\frac{\partial}{\partial\vec{r}}\left[\vec{A}\left(\vec{r},t\right)\dot{\vec{r}}\right]}-e\frac{\partial}{\partial\vec{r}}U\left(\vec{r}\right). \qquad (50)$$

We notice that from the vector formula

$$\dot{\vec{r}}\times\left[\frac{\partial}{\partial\vec{r}}\times\vec{A}\left(\vec{r},t\right)\right]=\underline{\frac{\partial}{\partial\vec{r}}\left[\dot{\vec{r}}\vec{A}\left(\vec{r},t\right)\right]}-\underline{\underline{\left(\dot{\vec{r}}\frac{\partial}{\partial\vec{r}}\right)\vec{A}\left(\vec{r},t\right)}} \qquad (51)$$

the first term of the expression (50) gets a form including the last term of equation (49). In this way, with the electric and magnetic fields,

$$\begin{aligned}\vec{E}\left(\vec{r},t\right)&=-\frac{\partial}{\partial\vec{r}}U\left(\vec{r}\right)-\frac{\partial}{\partial t}\vec{A}\left(\vec{r},t\right)\\\vec{B}\left(\vec{r},t\right)&=\frac{\partial}{\partial\vec{r}}\times\vec{A}\left(\vec{r},t\right),\end{aligned} \qquad (52)$$

the mechanical force (49) takes the form of Lorentz's force:

$$\vec{F}_e=e\vec{E}\left(\vec{r},t\right)+e\dot{\vec{r}}\times\vec{B}\left(\vec{r},t\right). \qquad (53)$$

From equations (52), we obtain the Faraday-Maxwell law of the of the electromagnetic induction, as a curl of the electric field induced by a time variation of the magnetic field,

$$\frac{\partial}{\partial\vec{r}}\times\vec{E}\left(\vec{r},t\right)=-\frac{\partial}{\partial t}\vec{B}\left(\vec{r},t\right), \qquad (54)$$

and the Gauss-Maxwell law of the null divergence of the magnetic field,

$$\frac{\partial}{\partial\vec{r}}\vec{B}\left(\vec{r},t\right)=0. \qquad (55)$$

With the gouge condition

$$\frac{\partial}{\partial\vec{r}}\vec{A}\left(\vec{r},t\right)=0, \qquad (56)$$

we obtain the Gauss-Maxwell law of the divergence of the electric field

$$\frac{\partial}{\partial\vec{r}}\vec{E}\left(\vec{r},t\right)=-\frac{\partial^2}{\partial\vec{r}^2}U\left(\vec{r}\right), \qquad (57)$$

with the charge density as a source of this field,

$$\rho\left(\vec{r}\right)=\varepsilon_0\frac{\partial}{\partial\vec{r}}\vec{E}\left(\vec{r},t\right)=-\varepsilon_0\frac{\partial^2}{\partial\vec{r}^2}U\left(\vec{r}\right). \qquad (58)$$

For a uniform distribution of the electric charge,

$$\frac{\partial}{\partial\vec{r}}\rho\left(\vec{r}\right)=\varepsilon_0\frac{\partial}{\partial\vec{r}}\left[\frac{\partial}{\partial\vec{r}}\vec{E}\left(\vec{r},t\right)\right]=0, \qquad (59)$$

from the curl of the Faraday-Maxwell law (54) we obtain the Laplacian of the electric field induced by a time variation of the curl of magnetic field,

$$\frac{\partial^2}{\partial\vec{r}^2}\vec{E}\left(\vec{r},t\right)-\frac{\partial}{\partial t}\frac{\partial}{\partial\vec{r}}\times\vec{B}\left(\vec{r},t\right)=0. \qquad (60)$$

On the other hand, we notice that an electromagnetic field, interacting with a quantum particle wave packet (40), with the cut-off velocity $c$, must propagate with the same velocity $c$ (**Fig. 3**), otherwise the particle and the field not interacting one another during the whole evolution.

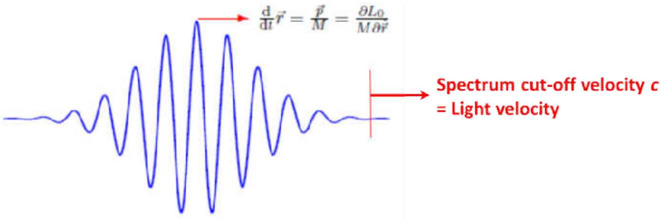


**Fig 3** Wave packet of a quantum particle with the cut-off velocity, interacting with an electromagnetic field propagating with the same velocity, called 'light velocity'.

In a material structure with a uniform mobile charge density $\rho$, a field propagating with a velocity $c$, and a decay rate $\gamma$ due to the interaction of this field with the electric charge, is described by a wave equation

$$\frac{\partial^2}{\partial\vec{r}^2}\vec{E}(\vec{r},t)-\frac{1}{c^2}\left[\frac{\partial^2}{\partial t^2}\vec{E}(\vec{r},t)+\gamma\frac{\partial}{\partial t}\vec{E}(\vec{r},t)\right]=0. \qquad (61)$$

From (60) and (61), by eliminating the Laplacian of the electric field and a time integration with an integration constant $\vec{j}_D(\vec{r})$, we obtain the Ampère-Maxwell law,

$$\frac{1}{\mu_0}\frac{\partial}{\partial\vec{r}}\times\vec{B}(\vec{r},t)=\varepsilon_0\gamma\vec{E}(\vec{r},t)+\varepsilon_0\frac{\partial}{\partial t}\vec{E}(\vec{r},t)+\vec{j}_D(\vec{r}), \qquad (62)$$

depending on and the light velocity $c=\frac{1}{\sqrt{\varepsilon_0\mu_0}}$. This equation can be considered of the more specific form with a total field variation in time,

$$\frac{1}{\mu_0}\frac{\partial}{\partial\vec{r}}\times\vec{B}(\vec{r},t)=\varepsilon_0\gamma\vec{E}(\vec{r},t)+\varepsilon_0\frac{\partial}{\partial t}\vec{E}(\vec{r},t)+\vec{j}_D(\vec{r})=\varepsilon_0\frac{\mathrm{d}}{\mathrm{d}t}\vec{E}(\vec{r},t)+\vec{j}_D(\vec{r}), \qquad (63)$$

including the component generated by a charge motion with the velocity $\dot{\vec{r}}$ induced by this field,

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{E}(\vec{r},t)=\frac{\partial}{\partial t}\vec{E}(\vec{r},t)+\underline{\left(\dot{\vec{r}}\frac{\partial}{\partial\vec{r}}\right)\vec{E}(\vec{r},t)}. \qquad (64)$$

For a charge velocity $\dot{\vec{r}}$ induced by this field,

$$\dot{\vec{r}}\times\vec{E}(\vec{r},t)=0, \qquad (65)$$

we obtain the last term of (64),

$$\frac{\partial}{\partial\vec{r}}\times\underbrace{\left[\dot{\vec{r}}\times\vec{E}(\vec{r},t)\right]}_{0}=\dot{\vec{r}}\left[\frac{\partial}{\partial\vec{r}}\vec{E}(\vec{r},t)\right]-\left(\dot{\vec{r}}\frac{\partial}{\partial\vec{r}}\right)\vec{E}(\vec{r},t)=\dot{\vec{r}}\frac{\rho(\vec{r},t)}{\varepsilon_0}-\underline{\left(\dot{\vec{r}}\frac{\partial}{\partial\vec{r}}\right)\vec{E}(\vec{r},t)}=0, \qquad (66)$$

as a function of the density $\rho(\vec{r},t)$. We define the electric current density of the mobile charge with a mobility $\mu$ in the considered material structure,

$$\vec{j}_\mu(\vec{r},t)=\rho(\vec{r},t)\dot{\vec{r}}=\rho(\vec{r},t)\mu\vec{E}(\vec{r},t)=\varepsilon_0\gamma\vec{E}(\vec{r},t), \qquad (67)$$

which means a decay rate of the electromagnetic field proportional to the mobility and the density of the electric charge,

$$\gamma=\frac{\mu}{\varepsilon_0}\rho(\vec{r},t). \qquad (68)$$

With (64), (66), and (67), equation (63) takes the conventional form of the Ampère-Maxwell law of the magnetic circuit, as a curl of the magnetic field induced by an electric current

$$\vec{j}(\vec{r},t)=\vec{j}_\mu(\vec{r},t)+\vec{j}_D(\vec{r}) \qquad (69)$$

and a time variation of the electric field,

$$\frac{1}{\mu_0}\frac{\partial}{\partial\vec{r}}\times\vec{B}(\vec{r},t)=\vec{j}(\vec{r},t)+\varepsilon_0\frac{\partial}{\partial t}\vec{E}(\vec{r},t). \qquad (70)$$

With (58), we find that this equation is in agreement with the charge conservation,

$$\frac{\partial}{\partial\vec{r}}\vec{j}(\vec{r},t)=-\frac{\partial}{\partial t}\rho(\vec{r},t). \qquad (71)$$

The total current (69) with the component $\vec{j}_D(\vec{r})$ may include the diffusion current, proportional to the gradient of the charge density, not taken into account in the equations (60) and (61), but reobtained as an integration constant.

We consider a unity vacuum impedance,

$$\zeta=\sqrt{\mu_0/\varepsilon_0}=1, \qquad \varepsilon_0=\mu_0=1/c, \qquad (72)$$

for the intensity of the magnetic field

$$\vec{H}(\vec{r},t)=\vec{B}(\vec{r},t)/\mu_0 \qquad (73)$$

as a function of the magnetic induction $\vec{B}(\vec{r},t)$, and the intensity of the electric field $\vec{E}(\vec{r},t)$. For a spherical geometry

$$\int_{V_\Sigma}\nabla\vec{E}\mathrm{d}^3\vec{r}=\oint_{\Sigma}\vec{E}\mathrm{d}^2\vec{r}=4\pi r^2\vec{E}\frac{\vec{r}}{r}=\frac{1}{\varepsilon_0}\int_{V_\Sigma}\rho\mathrm{d}^3\vec{r}, \qquad (74)$$

suggestinga normalization of the charge and current densities

$\rho$ and $\vec{j}=\rho\dot{\vec{r}}$,

$\rho\to 4\pi\varepsilon_0\rho$ (75)

$$\vec{j}=\rho\dot{\vec{r}}\to 4\pi\varepsilon_0\rho\dot{\vec{r}}=4\pi\vec{j}, \qquad (76)$$

the Maxwell equations (54), (70), (55), and (57) with (58), take the more symmetric form [15],

$$\frac{1}{c}\frac{\partial\vec{H}}{\partial t}=-\nabla\times\vec{E}, \qquad \nabla\vec{H}=0$$
$$\frac{1}{c}\frac{\partial\vec{E}}{\partial t}=\nabla\times\vec{H}-4\pi\vec{j}, \quad \nabla\vec{E}=4\pi\rho. \qquad (77)$$

In this representation of the Maxwell equations, the current density $\vec{j}=\rho\dot{\vec{r}}/c$ has the same dimension unit as the charge density $\rho$, and the electric field $\vec{E}$ the same dimension unit as the magnetic field $\vec{H}$, as the Coulomb law for an electric charge $q$ has the simpler form $\vec{E} = q\vec{r} / r^3$.

***Relativistic wave Equations***

For the wave function of a quantum particle (40), we consider the Lagrangian as a function of the Hamiltonian,

$$\psi\left(\vec{r},t\right) = \frac{1}{\left(2\pi h\right)^{3/2}} \int \varphi\left(\vec{P},t\right) e^{\frac{\mathrm{i}}{\hbar}\left\{\vec{P}\vec{r} - \left[\vec{P}\dot{\vec{r}} - H\left(\vec{P},\vec{r}\right)\right]t\right\}} \mathrm{d}^3\vec{P}, \tag{78}$$

as a solution of equation

$$-\mathrm{i}\hbar \frac{\partial}{\partial t} \psi\left(\vec{r},t\right) = \left[H\left(\vec{P},\vec{r}\right) - \vec{P}\dot{\vec{r}}\right] \psi\left(\vec{r},t\right). \tag{79}$$

It is interesting that with the canonical momentum

$$\vec{P} = \vec{p} + e\vec{A}\left(\vec{r},t\right) = -\mathrm{i}\hbar \frac{\partial}{\partial \vec{r}}, \tag{80}$$

the total derivative takes a form depending on this momentum,

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{\vec{r}} \frac{\partial}{\partial \vec{r}} = \frac{\partial}{\partial t} + \frac{\mathrm{i}}{\hbar} \vec{P}\dot{\vec{r}}. \tag{81}$$

With this expression, from (79) we obtain the relativistic quantum dynamic equation for a quantum particle of a form like the conventional Schrödinger equation,

$$-\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} \psi\left(\vec{r},t\right) = H\left(\vec{P},\vec{r}\right) \psi\left(\vec{r},t\right), \tag{82}$$

but with the total time derivative with changed sign instead of the partial derivative, and with the relativistic Hamiltonian (48) instead of the classical one as in the Dirac relativistic equation. We consider Dirac's Hamiltonian,

$$H\left(\vec{p},\vec{r}\right) = c\sqrt{M_0^2 c^2 + \vec{p}^2} + eU\left(\vec{r}\right) = c\left(\alpha_0 M_0 c + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3\right) + eU\left(\vec{r}\right), \tag{83}$$

With Dirac'soperators

$$\alpha_0 = \begin{pmatrix} \hat{1} & 0 \\ 0 & -\hat{1} \end{pmatrix}, \alpha_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \tag{84}$$

as functions of the Pauli spin operators,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{85}$$

and the anti-commutation relations,

$$\left\{\alpha_i, \alpha_j\right\} = 2\delta_{ij}, \qquad \left\{\sigma_i, \sigma_j\right\} = 2\delta_{ij}, \qquad \sigma_i \sigma_j = \mathrm{i}\delta_{ijk}\sigma_k. \tag{86}$$

With these operators, and the mechanical momentum as a function of the canonical momentum (80),

$$\vec{p} = -\mathrm{i}\hbar \frac{\partial}{\partial \vec{r}} - e\vec{A}\left(\vec{r},t\right), \tag{87}$$

for the wave function four-vector of a quantum particle in electromagnetic field,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}, \tag{88}$$

from the dynamic equation (82) we obtain the explicitequations

$$\begin{aligned}
&\left[\begin{aligned}&-\mathrm{i}\hbar\left(\frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}\right)\varphi_1\left(\vec{r},t\right) = \left(M_0c^2 + eU\left(\vec{r}\right)\right)\varphi_1\left(\vec{r},t\right)\\
&+c\left(-\mathrm{i}\hbar\frac{\partial}{\partial x} - eA_x\left(\vec{r},t\right)\right)\varphi_4\left(\vec{r},t\right) - \mathrm{i}c\left(-\mathrm{i}\hbar\frac{\partial}{\partial y} - eA_y\left(\vec{r},t\right)\right)\varphi_4\left(\vec{r},t\right) + c\left(-\mathrm{i}\hbar\frac{\partial}{\partial z} - eA_z\left(\vec{r},t\right)\right)\varphi_3\left(\vec{r},t\right),\end{aligned}\right.\\
&\left[\begin{aligned}&-\mathrm{i}\hbar\left(\frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}\right)\varphi_2\left(\vec{r},t\right) = \left(M_0c^2 + eU\left(\vec{r}\right)\right)\varphi_2\left(\vec{r},t\right)\\
&+c\left(-\mathrm{i}\hbar\frac{\partial}{\partial x} - eA_x\left(\vec{r},t\right)\right)\varphi_3\left(\vec{r},t\right) + \mathrm{i}c\left(-\mathrm{i}\hbar\frac{\partial}{\partial y} - eA_y\left(\vec{r},t\right)\right)\varphi_3\left(\vec{r},t\right) - c\left(-\mathrm{i}\hbar\frac{\partial}{\partial z} - eA_z\left(\vec{r},t\right)\right)\varphi_4\left(\vec{r},t\right),\end{aligned}\right.\\
&\left[\begin{aligned}&-\mathrm{i}\hbar\left(\frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}\right)\varphi_3\left(\vec{r},t\right) = \left(-M_0c^2 + eU\left(\vec{r}\right)\right)\varphi_3\left(\vec{r},t\right)\\
&+c\left(-\mathrm{i}\hbar\frac{\partial}{\partial x} - eA_x\left(\vec{r},t\right)\right)\varphi_2\left(\vec{r},t\right) - \mathrm{i}c\left(-\mathrm{i}\hbar\frac{\partial}{\partial y} - eA_y\left(\vec{r},t\right)\right)\varphi_2\left(\vec{r},t\right) + c\left(-\mathrm{i}\hbar\frac{\partial}{\partial z} - eA_z\left(\vec{r},t\right)\right)\varphi_1\left(\vec{r},t\right),\end{aligned}\right.\\
&\left[\begin{aligned}&-\mathrm{i}\hbar\left(\frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}\right)\varphi_4\left(\vec{r},t\right) = \left(-M_0c^2 + eU\left(\vec{r}\right)\right)\varphi_3\left(\vec{r},t\right)\\
&+c\left(-\mathrm{i}\hbar\frac{\partial}{\partial x} - eA_x\left(\vec{r},t\right)\right)\varphi_1\left(\vec{r},t\right) + \mathrm{i}c\left(-\mathrm{i}\hbar\frac{\partial}{\partial y} - eA_y\left(\vec{r},t\right)\right)\varphi_1\left(\vec{r},t\right) - c\left(-\mathrm{i}\hbar\frac{\partial}{\partial z} - eA_z\left(\vec{r},t\right)\right)\varphi_2\left(\vec{r},t\right).\end{aligned}\right.
\end{aligned} \tag{89}$$

Compared to Dirac's similar equations, obtained from the Schrödinger equation with Dirac's relativistic Hamiltonian, these equations include new terms depending on the particle momentum and velocity, which come from the new dynamic equation (82) including the total time derivative instead of the partial derivative. At the same time, from the relativistic wave function (78) for a particle with an energy $E$,

$$\psi\left(\vec{r},t\right) = \frac{1}{\left(2\pi h\right)^{3/2}} \int \varphi\left(\vec{P},t\right) e^{\frac{\mathrm{i}}{\hbar}\left\{\vec{P}\vec{r} - \left[\vec{P}\dot{\vec{r}} - H\left(\vec{P},\vec{r}\right)\right]t\right\}} \mathrm{d}^3\vec{P} = \frac{1}{\left(2\pi h\right)^{3/2}} \int \varphi\left(\vec{P},t\right) e^{\frac{\mathrm{i}}{\hbar}\left[\vec{P}\vec{r} + \left(E - \vec{P}\dot{\vec{r}}\right)t\right]} \mathrm{d}^3\vec{P}. \tag{90}$$

we obtain the time independent equation

$$H\left(\vec{P},\vec{r}\right) \psi\left(\vec{r}\right) = E\psi\left(\vec{r}\right), \tag{91}$$

similar to the conventional Schrödinger equation, but with the relativistic Hamiltonian (83).With Dirac's spin operators,

$$\vec{\alpha} = \left(\alpha_1, \alpha_2, \alpha_3\right), \tag{92}$$

and the mechanical momentum

$$\vec{p} = \left(p_1, p_2, p_3\right), \tag{93}$$

this equation takes the form

$$\left[c\left(\alpha_0 M_0 c + \vec{\alpha}\vec{p}\right) + eU\left(\vec{r}\right)\right] \psi\left(\vec{r},t\right) = E\psi\left(\vec{r},t\right). \tag{94}$$

With the Pauli spin vector

$$\vec{\sigma} = \left(\sigma_1, \sigma_2, \sigma_3\right), \tag{95}$$

for the wave function (88) with two vector components, this equation takes a form of two coupled two-dimensional equations,

$$\begin{aligned}
&\left[M_0c^2 + eU\left(\vec{r}\right)\right] \psi_1\left(\vec{r}\right) + c\vec{\sigma}\vec{p}\psi_2\left(\vec{r}\right) = E\psi_1\left(\vec{r}\right)\\
&\left[-M_0c^2 + eU\left(\vec{r}\right)\right] \psi_2\left(\vec{r}\right) + c\vec{\sigma}\vec{p}\psi_1\left(\vec{r}\right) = E\psi_2\left(\vec{r}\right).
\end{aligned} \tag{96}$$

By eliminating the coupling terms, we obtain a non-linear equation for the two-dimensional component $\psi_1(\vec{r})$ of the wave function (88),

$$\left[E+M_0c^2-eU(\vec{r})\right]\left[E-M_0c^2-eU(\vec{r})\right]\psi_1(\vec{r})=c^2(\vec{\sigma}\vec{p})^2\psi_1(\vec{r}), \tag{97}$$

and the same equation for the other two-dimensional component $\psi_2(\vec{r})$. With the mechanical momentum (93), and the Pauli spin vector (95) with the commutation relations (86), we obtain the coefficient of the right-hand side of this equation,

$$(\vec{\sigma}\vec{p})^2=(\sigma_1p_1+\sigma_2p_2+\sigma_3p_3)^2=\vec{p}^2+\mathrm{i}\vec{\sigma}(\vec{p}\times\vec{p}). \tag{98}$$

With the mechanical momentum (87), for the second term of this equation we obtain

$$\begin{aligned}(\vec{p}\times\vec{p})\psi&=\left[\left(-\mathrm{i}\hbar\nabla-e\vec{A}(\vec{r},t)\right)\times\left(-\mathrm{i}\hbar\nabla-e\vec{A}(\vec{r},t)\right)\right]\psi=\mathrm{i}\hbar e\left(\nabla\times\vec{A}+\vec{A}\times\nabla\right)\psi\\&=\mathrm{i}\hbar e\left[\nabla\times\left(\vec{A}\psi\right)+\vec{A}\times(\nabla\psi)\right]=\mathrm{i}\hbar e\left[\left(\nabla\times\vec{A}\right)\psi+\underline{(\nabla\psi)\times\vec{A}}+\underline{\vec{A}\times(\nabla\psi)}\right]\\&=\mathrm{i}\hbar e\left(\nabla\times\vec{A}\right)\psi=\mathrm{i}e\hbar\vec{B}\psi.\end{aligned} \tag{99}$$

Thus, the coefficient (98) takes a form depending on the magnetic field and the spin vector,

$$(\vec{\sigma}\vec{p})^2=\vec{p}^2-eh\vec{\sigma}\vec{B}. \tag{100}$$

With this coefficient, the dynamic equation (97) takes a form

$$\left(\frac{\vec{p}^2}{2M_0}-\vec{\mu}_s\vec{B}\right)\psi_1(\vec{r})=\left[\frac{E}{2M_0c^2}+\frac{1}{2}-\frac{e}{2M_0c^2}U(\vec{r})\right]\left[E-M_0c^2-eU(\vec{r})\right]\psi_1(\vec{r}), \tag{101}$$

depending on the spin magnetic moment,

$$\vec{\mu}_s=\frac{\hbar e}{2M_0}\vec{\sigma}=\frac{e}{M_0}\vec{s}=g_s\vec{s}, \tag{102}$$

proportional to the spin angular momentum

$$\vec{s}=\frac{1}{2}\vec{\sigma}, \tag{103}$$

with a coefficient called giro-magnetic ratio

$$g_s=e/M_0. \tag{104}$$

We notice that for the classical case of a small velocity, $E\approx M_0c^2$, and of a sufficiently weak electromagnetic field, $eU(\vec{r})<<M_0c^2$, $e\left|\vec{A}\right|<<\left|\vec{p}\right|$, equation (101) reduces to a the classical Schrödinger equation with spin interaction,

$$\left(\frac{\vec{p}^2}{2M_0}-\vec{\mu}_s\vec{B}+eU(\vec{r})\right)\psi_1(\vec{r})=E_c\psi_1(\vec{r}), \tag{105}$$

with the classical energy

$$E_c=E-M_0c^2. \tag{106}$$

On the other hand, with the mechanical momentum operator (87), which for a sufficiently weak magnetic field is

$$\vec{p}=-\mathrm{i}\hbar\frac{\partial}{\partial\vec{r}}, \tag{107}$$

we find that the relativistic dynamic equation (101) includes only an orbital angular momentum $l_j$ of the particle in the electric potential $U(\vec{r})$, which satisfies the commutation relations

$$\left[p_i,l_j\right]=\mathrm{i}\hbar\delta_{ijk}p_k, \tag{108}$$

as an additional term to the spin angular momentum in the total angular momentum

$$\vec{j}=\vec{l}+\vec{s}. \tag{109}$$

We consider the condition of commutation of a component $j_3$ with the particle Hamiltonian

$$H(\vec{p},\vec{r})=c\left(\alpha_0M_0c+\alpha_1p_1+\alpha_2p_2+\alpha_3p_3\right), \tag{110}$$

$$\left[H,j_3\right]=\left[H,l_3+s_3\right]=0. \tag{111}$$

From the equality of the two terms

$$\begin{aligned}&\left[H,s_3\right]=c\left(\left[\alpha_1,s_3\right]p_1+\left[\alpha_2,s_3\right]p_2+\left[\alpha_3,s_3\right]p_3+\left[\alpha_4,s_3\right]M_0c\right)\\&-\left[H,l_3\right]=c\left(-\alpha_1\left[p_1,l_3\right]-\alpha_2\left[p_2,l_3\right]-\alpha_3\left[p_3,l_3\right]\right)=\mathrm{i}\hbar c\left(\alpha_1p_2-\alpha_2p_1\right),\end{aligned} \tag{112}$$

with the commutation relations (108), we obtain equations for the considered spin component,

$$\left[\alpha_1,s_3\right]=-\mathrm{i}\hbar\alpha_2,\quad\left[\alpha_2,s_3\right]=\mathrm{i}\hbar\alpha_1,\quad\left[\alpha_3,s_3\right]=\left[\alpha_4,s_3\right]=0. \tag{113}$$

With the anti-commutation/normalization relations (86),

$$\alpha_2\alpha_1=-\alpha_1\alpha_2,\ \alpha_1^{\ 2}=\alpha_2^{\ 2}=1, \tag{114}$$

we obtain the considered component of the spin angular momentum (103),

$$s_3=-\mathrm{i}\frac{\hbar}{2}\alpha_1\alpha_2=-\mathrm{i}\frac{\hbar}{2}\begin{pmatrix}0&\sigma_1\\\sigma_1&0\end{pmatrix}\begin{pmatrix}0&\sigma_2\\\sigma_2&0\end{pmatrix}=-\mathrm{i}\frac{\hbar}{2}\begin{pmatrix}\sigma_1\sigma_2&0\\0&\sigma_1\sigma_2\end{pmatrix}=\frac{\hbar}{2}\begin{pmatrix}\sigma_3&0\\0&\sigma_3\end{pmatrix}. \tag{115}$$

This means that just in the relativistic case, but for a sufficiently weak magnetic field, the total angular momentum has the simple form (109), as the sum of the orbital angular momentum with the proper angular momentum (the spin) of the extrinsic matter of a quantum particle. Otherwise, a significant additional angular momentum arise due to the magnetic field, from the vector potential $\vec{A}(\vec{r},t)$ in the kinetic tem of equation (101) with the momentum (87).

***Quantum Particle in Gravitational field and black hole***

In a gravitational field, the physical hyper surface $\left(x^0=ct,x^1,x^2,x^3\right)$ gets a curvature. We consider a time constant gravitational field (null space-time metric elements, time independent metric elements), and determine the metric tensorina reference system of spherical coordinates $(ct,r,\theta,\varphi)$, from Einstein's law of gravitation of a null Ricci tensor,

$$R_{\mu\nu}=\Gamma^{\alpha}_{\mu\alpha,\nu}-\Gamma^{\alpha}_{\mu\nu,\alpha}+\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\beta\nu}-\Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\beta\alpha}=0. \tag{116}$$

In this case, for the metric tensor elements one can consider a form dependingon two functions $u(r)$ and $v(r)$ [15],

$$\begin{aligned} &g_{00}=e^{2u(r)}, \quad g_{11}=-e^{2v(r)}, \quad g_{22}=-r^2, \quad g_{33}=-r^2\sin^2\theta \\ &g^{00}=e^{-2u(r)}, \quad g^{11}=-e^{-2v(r)}, \quad g^{22}=-r^{-2}, \quad g^{33}=-r^{-2}\sin^{-2}\theta. \end{aligned} \tag{117}$$

From the diagonal matrix elements of equation (116), we obtain the system of equations:

$$\begin{aligned} &R_{00}=\left[-u''(r)-u'^2(r)+u'(r)v'(r)-2r^{-1}u'(r)\right]e^{2u(r)-2v(r)}=0 \\ &R_{11}=u''(r)+u'^2(r)-u'(r)v'(r)-2r^{-1}v'(r)=0 \\ &R_{22}=-1+\left[1+ru'(r)-rv'(r)\right]e^{-2v(r)}=0 \\ &R_{33}=\sin^2\theta R_{22}=0. \end{aligned} \tag{118}$$

From the first two equations, we obtain equation

$$u'(r)+v'(r)=0, \tag{119}$$

with a solution

$$u(r)+v(r)=0. \tag{120}$$

With these two equations, from the third equation (118) we obtain the differential equations

$$\begin{aligned} &\left[1+2ru'(r)\right]e^{2u(r)}=\left[re^{2u(r)}\right]'=1 \\ &\left[1-2rv'(r)\right]e^{-2v(r)}=\left[re^{-2v(r)}\right]'=1, \end{aligned} \tag{121}$$

with the solutions

$$\begin{aligned} &e^{2u(r)}=1-\frac{2m}{r} \\ &e^{2v(r)}=\left(1-\frac{2m}{r}\right)^{-1}, \end{aligned} \tag{122}$$

depending on an integration constant $m$. For the determination of the two functions $u(r)$ and $v(r)$ we used only a combination (119) of the first two equations (118), and the third equation (118). It is interesting that the derivative of the first equation (121),

$$\left[re^{2u(r)}\right]''=2\left[ru''(r)+2ru'^2(r)+2u'(r)\right]e^{2u(r)}=0, \tag{123}$$

is in agreement with the second equation (118) with the relation (119),

$$R_{11}=\frac{ru''(r)+2ru'^2(r)+2u'(r)}{r}=0, \tag{124}$$

which means that (122) is a solution of the whole system of equations (118). In this way, the matrix elements (117) take the form of the Schwarzschild solution of the metric tensor for a constant gravitational field in spherical coordinates,

$$\begin{aligned} &g_{00}=1-\frac{2m}{r}, \quad g_{11}=-\left(1-\frac{2m}{r}\right)^{-1}, \quad g_{22}=-r^2, \quad g_{33}=-r^2\sin^2\theta \\ &g^{00}=\left(1-\frac{2m}{r}\right)^{-1}, \quad g^{11}=-\left(1-\frac{2m}{r}\right), \quad g^{22}=-r^{-2}, \quad g^{33}=-r^{-2}\sin^{-2}\theta, \end{aligned} \tag{125}$$

which means a time-space interval in a central gravitational field of the form

$$\mathrm{d}s^2=\left(1-\frac{2m}{r}\right)c^2\mathrm{d}t^2-\left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2-r^2\mathrm{d}\theta^2-r^2\sin^2\theta\mathrm{d}\varphi^2. \tag{126}$$

For a quantum particle in gravitational field, we consider wave functions of the form (34) and (37) with a Lagrangian of the form (35), with a time-space diagonalization of the metric tensor,

$$\begin{aligned} \psi(x^i,t)&=\frac{1}{(2\pi\hbar)^{3/2}}\int\varphi(P_j,t)e^{\frac{i}{\hbar}(P_jx^j-Lt)}\mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P_3 \\ &=\frac{1}{(2\pi\hbar)^{3/2}}\int\varphi(P_j,t)e^{\frac{i}{\hbar}\left[P_jx^j+M_0c^2\sqrt{g_{00}+g_{ij}\dot{x}^i\dot{x}^j}t\right]}\mathrm{d}P_1\mathrm{d}P_2\mathrm{d}P_3 \\ \varphi(P_j,t)&=\frac{1}{(2\pi\hbar)^{3/2}}\int\psi(x^i,t)e^{-\frac{i}{\hbar}(P_jx^j-Lt)}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 \\ &=\frac{1}{(2\pi\hbar)^{3/2}}\int\psi(x^i,t)e^{-\frac{i}{\hbar}\left[P_jx^j+M_0c^2\sqrt{g_{00}+g_{ij}\dot{x}^i\dot{x}^j}t\right]}\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3, \end{aligned} \tag{127}$$

and the momentum (36),

$$P_j=\frac{\partial L}{c\partial\dot{x}^j}=-\frac{1}{2}M_0cg_{ij}\dot{x}^i. \tag{128}$$

With this expression and the fundamental relation (26), we obtain the group velocity of a quantum particle wave packet (127),

$$\frac{\mathrm{d}}{\mathrm{d}t}x^j=\frac{\partial\left(-M_0c^2\sqrt{g_{00}+g_{ij}\dot{x}^i\dot{x}^j}\right)}{\partial\left(-\frac{1}{2}M_0cg_{ij}\dot{x}^i\right)}=\frac{c\dot{x}^j}{\sqrt{g_{00}+g_{\alpha j}\dot{x}^\alpha\dot{x}^j}}=c\dot{x}^j. \tag{129}$$

With this velocity, we define the Hamiltonian of the quantum particle

$$H=cP_j\dot{x}^j-L\left(x^j,\dot{x}^j,t\right). \tag{130}$$

With the definition (128) of the momentum and the group velocity of the second wave packet (127) for the time derivative of this momentum, from the differential of the Hamiltonian (130),

$$\begin{aligned} \mathrm{d}H&\doteq\frac{\partial H}{\partial P_j}\mathrm{d}P_j+\frac{\partial H}{\partial x^j}\mathrm{d}x^j+\frac{\partial H}{\partial t}\mathrm{d}t \\ &=c\dot{x}^j\mathrm{d}P_j+cP_j\mathrm{d}\dot{x}^j-\frac{\partial L}{\partial x^j}\mathrm{d}x^j-\frac{\partial L}{\partial\dot{x}^j}\mathrm{d}\dot{x}^j-\frac{\partial L}{\partial t}\mathrm{d}t \\ &=c\dot{x}^j\mathrm{d}P_j+\underline{cP_j\mathrm{d}\dot{x}^j}-c\dot{P}_j\mathrm{d}x^j-\underline{cP_j\mathrm{d}\dot{x}^j}-\frac{\partial L}{\partial t}\mathrm{d}t, \end{aligned} \tag{131}$$

we obtain the Hamilton equations

$$\frac{\mathrm{d}x^j}{\mathrm{d}t} = \frac{\partial H}{\partial P_j}$$
$$\frac{\mathrm{d}P_j}{\mathrm{d}t} = -\frac{\partial H}{\partial x^j} \tag{132}$$
$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

With these equations, for a conservative system,

$$\frac{\partial L}{\partial t} = 0, \tag{133}$$

we obtain the Hamiltonian conservation,

$$\frac{\mathrm{d}}{\mathrm{d}t} H\left(P_j, x^j\right) = \frac{\partial H}{\partial P_j}\frac{\mathrm{d}P_j}{\mathrm{d}t} + \frac{\partial H}{\partial x^j}\frac{\mathrm{d}x^j}{\mathrm{d}t} = -\frac{\partial H}{\partial P_j}\frac{\partial H}{\partial x^j} + \frac{\partial H}{\partial x^j}\frac{\partial H}{\partial P_j} = 0, \tag{134}$$

with an eigenvalue equal to the energy of the quantum particle,

$$H\left(P_j, x^j\right) = E. \tag{135}$$

If we consider an atom emitting light in a central gravitational field, from (126) we obtain the frequency of this light as a function of its distance $r$ from the gravitational center,

$$\frac{1}{\Delta t} = \sqrt{1 - \frac{r_0}{r}}\frac{c}{\Delta s}, \tag{136}$$

and the constant

$$r_0 = 2m. \tag{137}$$

For a decreasing $r$, the frequency (136) describes a redshift of the electromagnetic radiation – at the boundary of a black hole, $r = r_0$, no light is emitted. For a better description of a black hole, we reconsider the geodesic equation for the general relativistic case,

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}s^2} = -\Gamma^\alpha_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}, \tag{138}$$

of a radial motion,

$$\frac{\mathrm{d}x^2}{\mathrm{d}s} = \frac{\mathrm{d}x^3}{\mathrm{d}s} = 0. \tag{139}$$

With a diagonal metric tensor, we obtain the geodesic equations

$$\begin{aligned}\frac{\mathrm{d}^2x^0}{\mathrm{d}s^2} &= -\Gamma^0_{01}\frac{\mathrm{d}x^0}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s} - \Gamma^0_{10}\frac{\mathrm{d}x^1}{\mathrm{d}s}\frac{\mathrm{d}x^0}{\mathrm{d}s} = -2g^{00}\Gamma_{001}\frac{\mathrm{d}x^0}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s}\\ &= -g^{00}\left(g_{00,1} + g_{01,0} - g_{01,0}\right)\frac{\mathrm{d}x^0}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s} = -g^{00}g_{00,1}\frac{\mathrm{d}x^0}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s} = -g^{00}\frac{\mathrm{d}g_{00}}{\mathrm{d}s}\frac{\mathrm{d}x^0}{\mathrm{d}s}\\ \frac{\mathrm{d}^2x^1}{\mathrm{d}s^2} &= -\Gamma^1_{01}\frac{\mathrm{d}x^0}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s} - \Gamma^1_{10}\frac{\mathrm{d}x^1}{\mathrm{d}s}\frac{\mathrm{d}x^0}{\mathrm{d}s} = -2g^{11}\Gamma_{101}\frac{\mathrm{d}x^0}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s}\\ &= -g^{11}\left(g_{10,1} + g_{11,0} - g_{10,1}\right)\frac{\mathrm{d}x^0}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s} = -g^{11}g_{11,0}\frac{\mathrm{d}x^0}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s} = -g^{11}\frac{\mathrm{d}g_{11}}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s}\end{aligned} \tag{140}$$

which, with the relations

$$g^{00} = \frac{1}{g_{00}}, \qquad g^{11} = \frac{1}{g_{11}}, \tag{141}$$

take an integrable form

$$\begin{aligned} g_{00}\frac{\mathrm{d}^2x^0}{\mathrm{d}s^2} + \frac{\mathrm{d}g_{00}}{\mathrm{d}s}\frac{\mathrm{d}x^0}{\mathrm{d}s} &= 0\\ g_{11}\frac{\mathrm{d}^2x^1}{\mathrm{d}s^2} + \frac{\mathrm{d}g_{11}}{\mathrm{d}s}\frac{\mathrm{d}x^1}{\mathrm{d}s} &= 0\end{aligned} \tag{142}$$

for the velocities

$$v^0 = \frac{\mathrm{d}x^0}{\mathrm{d}s}, \qquad v^1 = \frac{\mathrm{d}x^1}{\mathrm{d}s}. \tag{143}$$

With the integration constants $k_0$ and $k_1$, we obtain

$$\begin{aligned} g_{00}v^0 &= k_0\\ g_{11}v^1 &= k_1.\end{aligned} \tag{144}$$

From the first equation with the Schwarzschild solution (125), we obtain the velocity

$$v^0 = \frac{k_0}{1 - \frac{2m}{r}}, \tag{145}$$

as a function of the integration constant $k_0$. We notice that, with the fundamental equation (26) and the Schwarzschild solution (125), we can eliminate the other integration constant

$$k_1, \quad \begin{aligned} g_{00}{v^0}^2 + g_{11}{v^1}^2 &= 1\\ g_{00}g_{11} &= -1.\end{aligned} \tag{146}$$

With (144), the first equation (146) becomes

$$k_0v^0 + k_1v^1 = 1, \tag{147}$$

as by multiplication with $g_{00}$, with the second equation (146) and (125), it is of the form

$${k^0}^2 - {v^1}^2 = 1 - \frac{2m}{r}. \tag{148}$$

We obtain the velocity $v^1$ of an approaching particle, as a function of the same integration constant $k_0$,

$$v^1 = -\left({k_0}^2 + \frac{2m}{r} - 1\right)^{1/2}, \tag{149}$$

as the time velocity of this particle is

$$\frac{\mathrm{d}r}{\mathrm{d}t} = c\frac{\mathrm{d}x^1}{\mathrm{d}x^0} = c\frac{v^1}{v^0} = -\frac{c}{k_0}\left({k_0}^2 + \frac{2m}{r} - 1\right)^{1/2}\left(1 - \frac{2m}{r}\right). \tag{150}$$

According to this theory, for a particle approaching a black hole from the outside, $r > 2m$, its velocity decreases, becoming null at the boundary of this black hole – this particle never enters the black hole. On the other hand, for a particle approaching the boundary of a black hole from the inside, $r < 2m$, its velocity also decreases – a particle never exits from the black hole. In other words, according to (150), a particle moving in the proper system of a black hole seems to infinitely delay at the boundary of this black hole. However, this reasoning is not exactly true for the quantum dynamics, where a differential element of matter is not lonely approaching the boundary of a black hole, but always as a part of the matter distribution of a quantum particle, as it is described by the wave functions (127), the other differential elements perturbing the gravitational field considered in this theory. More than that, the case of a single particle approaching the boundary of a black hole is not a realistic one. A large number of quantum particles significantly perturb the gravitational field, making the boundary of black hole penetrable from the outside and from the inside – absorption and evaporation processes of a black hole are always present. From the geodesic equation of a particle with a trajectory as a function of a parameter $\tau$,

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = -\Gamma^\mu_{\alpha\beta} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}, \qquad (151)$$

with the velocities $c\dot{x}^\alpha, c\dot{x}^\beta$, we obtain the particle acceleration

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -c^2 \Gamma^i_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta, \qquad (152)$$

which, with the Christoffel symbol as a function of the metric tensor elements, is

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\frac{1}{2} c^2 g^{i\lambda} \left( g_{\lambda\alpha,\beta} + g_{\lambda\beta,\alpha} - g_{\alpha\beta,\lambda} \right) \dot{x}^\alpha \dot{x}^\beta. \qquad (153)$$

This expression takes a simpler form for a small velocity compared to the light velocity, $\dot{x}^i << \dot{x}^0$, $\dot{x}^0 \approx 1$,

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\frac{1}{2} c^2 g^{i\lambda} \left( g_{\lambda 0,0} + g_{\lambda 0,0} - g_{00,\lambda} \right), \qquad (154)$$

which, for a diagonal metric tensor and a constant gravitational field is

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = \frac{1}{2} c^2 g^{ii} g_{00,i}. \qquad (155)$$

It is interesting that this assumption is also reasonable at the boundary of a black hole, where, according to (150), the velocity is null, a quantum particle penetrating this boundary only due to some perturbations of the gravitational field induced by this particle, or by other quantum particles, always present in the realistic cases. With the Schwarzschild metric element (125), of the form

$$g_{00} = 1 + 2V, \qquad (156)$$

as a function of the Newtonian potential

$$V = -\frac{m}{r}, \qquad (157)$$

from (155) we obtain the particle accelerations

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = c^2 g^{ii} V_{,i} = -c^2 g^{ii} \left( \frac{m}{r} \right)_{,i}. \qquad (158)$$

With Schwarzschild matrix elements (125), these accelerations take a form

$$\begin{aligned} &\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\left(1 - \frac{2m}{r}\right)\frac{mc^2}{r^2} = -\left(1 - \frac{2GM}{c^2 r}\right) G\frac{M}{r^2} \\ &\frac{\mathrm{d}^2 \theta}{\mathrm{d}t^2} = 0 \\ &\frac{\mathrm{d}^2 \varphi}{\mathrm{d}t^2} = 0, \end{aligned} \qquad (159)$$

depending onthe constant of gravitation $G = 6.67259 \times 10^{-11}\ \mathrm{Kg}^{-1}\mathrm{m}^3\mathrm{s}^{-2}$ and the mass $M$ generating the gravitational potential. For

$$r >> r_0 = 2m = \frac{2GM}{c^2} = 2 \times 7.4243 \times 10^{-28} M\ \mathrm{m},$$

equations (159) describe the dynamics of a quantum particle in a Newtonian potential

$$U_0(r) = -G\frac{M}{r}. \qquad (160)$$

At the same time, we find that for a black hole, with a sufficiently strong gravitational field for confining the mass $M$ in a sphere with the radius

$$r_M < r_0 = 2m, \qquad (161)$$

the attraction acceleration (159) for $r > r_0$,for $r < r_0$ turns out into a repulsion one, as for $r = r_0$ it is null. We notice that for our planet with the mass $5.9722 \times 10^{24}\ \mathrm{Kg}$ which leads to $r_0 = 0.0089\ \mathrm{m}$, with the radius $r'_M = 6378\ \mathrm{m}$ at equator, and $r''_M = 6357\ \mathrm{m}$ at a pole, the condition (161) is far from being satisfied.

***Quantum Particle in a Gravitational wave and the Graviton spin***

We consider a quantum particle described by the wave functions

$$\begin{aligned} &\psi\left(x^{\mathrm{i}}, t\right) = \frac{1}{(2\pi\hbar)^{3/2}} \int \varphi\left(P_j, t\right) e^{\frac{\mathrm{i}}{\hbar}\left[P_j x^{\mathrm{j}} - L\left(x^\alpha, \dot{x}^\alpha\right)t\right]} \mathrm{d}P_1 \mathrm{d}P_2 \mathrm{d}P_3 \\ &\varphi\left(P_j, t\right) = \frac{1}{(2\pi\hbar)^{3/2}} \int \psi\left(x^{\mathrm{i}}, t\right) e^{-\frac{\mathrm{i}}{\hbar}\left[P_j x^{\mathrm{j}} - L\left(x^\alpha, \dot{x}^\alpha\right)t\right]} \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3, \end{aligned} \qquad (162)$$

with the Lagrangian

$$L\left(x^\alpha, \dot{x}^\alpha\right) = -M_0 c^2 \sqrt{g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta}, \qquad (163)$$

under the action of a gravitational wave described by the d'Alembert equation,

$$g^{\mu\nu}g_{\rho\sigma,\mu\nu}=0, \quad (164)$$

as, with a time-space diagonalization of the metric tensor, $g_{j0}=0$, the canonical momentum is

$$P_j=\frac{\partial L}{c\partial \dot{x}^j}=-\frac{1}{2}M_0 c g_{ij}\dot{x}^i . \quad (165)$$

From the time derivative of this momentum as a group velocity of the wave packet (162), with the fundamental relation (26),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}P_j=c\dot{P}_j&=-\frac{1}{2}M_0c^2 g_{ij,k}\dot{x}^k\dot{x}^i-\frac{1}{2}M_0c^2 g_{ij}\ddot{x}^i=\frac{\partial L}{\partial x^j}\\ &=-M_0c^2\frac{\dot{x}^\alpha\dot{x}^\beta g_{\alpha\beta,j}}{2\sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}=-\frac{1}{2}M_0c^2 g_{\alpha\beta,j}\dot{x}^\alpha\dot{x}^\beta ,\end{aligned} \quad (166)$$

and the geodesic equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\dot{x}^\mu=\ddot{x}^\mu=-\Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta , \quad (167)$$

we obtain the dynamic equation

$$g_{ij,k}\dot{x}^i\dot{x}^k-g_{ij}\Gamma^i_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta=g_{\alpha\beta,j}\dot{x}^\alpha\dot{x}^\beta . \quad (168)$$

With the second kind Christoffel symbol,

$$\Gamma^i_{\alpha\beta}=g^{i\mu}\Gamma_{\mu\alpha\beta}=g^{i\mu}\frac{1}{2}\left(g_{\mu\alpha,\beta}+g_{\mu\beta,\alpha}-g_{\alpha\beta,\mu}\right)=g^{ij}\frac{1}{2}\left(g_{j\alpha,\beta}+g_{j\beta,\alpha}-g_{\alpha\beta,j}\right), \quad (169)$$

we obtain a dynamic equation depending only on the derivatives of the metric tensor,

$$g_{ij,k}\dot{x}^i\dot{x}^k-\frac{1}{2}\left(g_{j\alpha,\beta}+g_{j\beta,\alpha}+g_{\alpha\beta,j}\right)\dot{x}^\alpha\dot{x}^\beta=0 . \quad (170)$$

By a diagonalization of the time dependent component of the metric tensor,

$$g_{ji,0}=0, \quad (171)$$

from the dynamic equation (170) we obtain

$$-\frac{1}{2}\left(-g_{ji,k}+g_{jk,i}+g_{ik,j}\right)\dot{x}^i\dot{x}^k=\frac{1}{2}g_{00,j}\dot{x}^0\dot{x}^0=\frac{1}{2}g_{00,j} . \quad (172)$$

Since the gravitational waves are generally generated by heavy cosmic objects, in these equations we considered a small velocity, $\dot{x}^0=1$, as the geodesic equation (167) takes the form

$$\ddot{x}^\mu=-\Gamma^\mu_{ij}\dot{x}^i\dot{x}^j-\Gamma^\mu_{00}\dot{x}^0\dot{x}^0=-\Gamma^\mu_{ij}\dot{x}^i\dot{x}^j-\Gamma^\mu_{00} . \quad (173)$$

At the same time, in equation (172) we distinguish the first kind Christioffel symbol,

$$-\Gamma_{kij}\dot{x}^i\dot{x}^k=\frac{1}{2}g_{00,j} . \quad (174)$$

By multiplying equation (174) with $\dot{x}^j$, with the geodesic equation (173) and the time-space diagonalization of the metric tensor, we obtain the acceleration of a quantum particle in a gravitational field,

$$-g_{\mu k}\Gamma^\mu_{ij}\dot{x}^i\dot{x}^j\dot{x}^k=\dot{x}^k g_{\mu k}\left(\ddot{x}^\mu+\Gamma^\mu_{00}\right)=\dot{x}^k\left(\ddot{x}_k-\frac{1}{2}g_{00,k}\right)=\frac{1}{2}g_{00,j}\dot{x}^j , \quad (175)$$

of the form of a gradient of the metric element $g_{00}$ which, in this way, appears as a gravitational potential,

$$\ddot{x}_k=g_{00,k} , \quad (176)$$

Or

$$\ddot{x}^k=g^{kj}g_{00,j} . \quad (177)$$

As a solution of the gravitational wave equation (164), we consider a metric tensor of the second order in coordinates,

$$g_{\rho\sigma}=u_{\rho\sigma}l_{\mu\nu}x^\mu x^\nu , \quad (178)$$

proportional to an amplitude tensor $u_{\rho\sigma}$, and a polarization tensor $l_{\mu\nu}$ which, according to the equation of propagation (164), satisfies the normalization condition

$$l^\nu_\nu=0 . \quad (179)$$

With the solution (178) for the metric tensor in a gravitational wave, the dynamic equation (177) is

$$\ddot{x}^k=u_{00}l^k_\mu x^\mu . \quad (180)$$

For a gravitational field oscillating in the direction $x^1$,

$$l^0_0=1,\quad l^1_1=-1,\quad l^2_2=l^3_3=0,\qquad l^0_0+l^1_1+l^2_2+l^3_3=0 , \quad (181)$$

the dynamic equation (180) takes the form of harmonic oscillation:

$$\ddot{x}^1=-u_{00}x^1 . \quad (182)$$

With the metric elements for a weak gravitational field,

$$g_{00}=1,\quad g_{11}=-1,\quad g_{22}=-1,\quad g_{33}=-1 , \quad (183)$$

from (181) we obtain the polarization elements

$$l_{00}=g_{00}l^0_0=1,\quad l_{11}=g_{11}l^1_1=1,\quad l_{22}=g_{22}l^2_2=0,\quad l_{33}=g_{33}l^3_3=0 , \quad (184)$$

as the whole dynamics of the metric tensor $g_{\rho\sigma}$ is described by the coordinate dependence (178) and the amplitude tensor. Since $l_{\mu\nu}x^\mu x^\nu$ in (178) is a scalar, the amplitude tensor is proportional to the metric tensor:

$$u_{\rho\sigma}=ug_{\rho\sigma},\quad u^{\rho\sigma}=ug^{\rho\sigma} , \quad (185)$$

where $u$ is the 'total', or the 'scalar' amplitude. This amplitude can be considered as a function of the amplitude matrix elements,

$$g_{\mu\rho}u^{\rho\sigma}=u^\sigma_\mu=ug_{\mu\rho}g^{\rho\sigma}=ug^\sigma_\mu . \quad (186)$$

By the contraction of the indexes $\sigma$ and $\mu$, we obtain

$$u^\mu_\mu=ug^\mu_\mu=4u . \quad (187)$$

On the other hand, from (185) with (183) we obtain

$$u_{00} = ug_{00} = u \,, \tag{188}$$

as the dynamic equation (182) takes the form

$$\ddot{x}^1 = -ux^1 \,. \tag{189}$$

On the other hand, we notice that the gravitational wave equation (164) has a solution of the first order in coordinates,

$$g_{\rho\sigma} = u_{\rho\sigma} l_\mu x^\mu \,, \tag{190}$$

we consider with a normalization condition similar to (179),

$$l^\nu l_\nu = 0 \,. \tag{191}$$

With this solution, the dynamic equation (177) with (188) becomes

$$\ddot{x}^k = ul^k \,, \tag{192}$$

which means an acceleration of the quantum particle under the action of the gravitational wave. By multiplying equation (186) with the polarization vector $l_\sigma$, we obtain equations for the amplitude matrix elements as functions of the total amplitude,

$$l_\sigma u_\mu^\sigma = l_\mu u \,. \tag{193}$$

With the metric elements for a weak gravitational wave

$$g^{00} = 1, \quad g^{11} = g^{22} = g^{33} = -1 \,, \tag{194}$$

propagating in the direction $x^3$, we obtain the polarization vectors

$$\begin{aligned} &l_0 = 1, \quad l_1 = l_2 = 0, \quad l_3 = -1 \\ &l^0 = 1, \quad l^1 = l^2 = 0, \quad l^3 = 1, \end{aligned} \tag{195}$$

which satisfy the normalization equations (191), as the metric dynamics is described by the amplitudes. With these vectors and the metric elements (194), equations (193) take the form

$$\begin{aligned} u_0^0 - u_0^3 &= g^{00}u_{00} - g^{33}u_{30} = u_{00} + u_{30} = u \\ u_1^0 - u_1^3 &= g^{00}u_{01} - g^{33}u_{31} = u_{01} + u_{31} = 0 \\ u_2^0 - u_2^3 &= g^{00}u_{02} - g^{33}u_{32} = u_{02} + u_{32} = 0 \\ u_3^0 - u_3^3 &= g^{00}u_{03} - g^{33}u_{33} = u_{03} + u_{33} = -u \,. \end{aligned} \tag{196}$$

From the first and the last equations, with (187) and the metric elements (194), we obtain

$$\underline{u_{00} - u_{33}} = 2u = \frac{1}{2}u_\mu^\mu = \frac{1}{2}\left(g^{00}u_{00} + g^{11}u_{11} + g^{22}u_{22} + g^{33}u_{33}\right) = \frac{1}{2}\left(\underline{u_{00}} - u_{11} - u_{22} - \underline{u_{33}}\right), \tag{197}$$

which yields

$$u_{11} + u_{22} = -\left(u_{00} - u_{33}\right) = -2u \,, \tag{198}$$

and

$$2u_{03} = -\left(u_{00} + u_{33}\right). \tag{199}$$

With these equations, the transform equation

$$u^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} u_{\mu\nu} = g^{\alpha\alpha} g^{\beta\beta} u_{\alpha\beta} \,, \tag{200}$$

and the second and the third equations (196), we calculate the invariant

$$\begin{aligned} I_u &= u_{\alpha\beta}u^{\alpha\beta} - 2u^2 \\ &= u_{00}{}^2 + u_{11}{}^2 + u_{22}{}^2 + u_{33}{}^2 - 2\underline{u_{01}}{}^2 - 2\underline{\underline{u_{02}}}{}^2 \underline{\underline{-2u_{03}}}{}^2 + 2u_{12}{}^2 + 2\underline{u_{13}}{}^2 + 2\underline{\underline{u_{23}}}{}^2 - \frac{1}{2}\left(u_{00} - u_{33}\right)^2 \\ &= u_{11}{}^2 + u_{22}{}^2 + 2u_{12}{}^2 + \underbrace{u_{00}{}^2 + u_{33}{}^2 \underline{\underline{-\frac{1}{2}\left(u_{00}{}^2 + u_{33}{}^2\right) - u_{00}u_{33}}}} \underbrace{-\frac{1}{2}\left(u_{00} - u_{33}\right)^2} \\ &= u_{11}{}^2 + u_{22}{}^2 + 2u_{12}{}^2 = \frac{1}{2}\left[\underline{\left(u_{11} - u_{22}\right)^2} + \underbrace{\left(u_{11} + u_{22}\right)^2}_{\underline{\underline{4u^2}}}\right] + \underline{2u_{12}{}^2}, \end{aligned} \tag{201}$$

where we distinguish a term describing a proper dynamics,

$$I_0 = \frac{1}{2}\left(u_{11} - u_{22}\right)^2 + 2u_{12}{}^2 \,, \tag{202}$$

and a term $2u^2$, describing the action of the gravitational wave on the quantum particle,

$$I_u = I_0 + 2u^2 \,. \tag{203}$$

The invariant (202) describes a rotation of the amplitude tensor $u_{\mu\nu}$. For a description of this rotation, we define the operator $R$ of the rotation of a vector $A = \left(A^1, A^2\right)$ with an angle $-\dfrac{\pi}{4}$ in a plane $(x, y)$ (**Fig. 4**),

$$\begin{aligned} RA^1 &= A^2 \\ RA^2 &= -A^1 \,. \end{aligned} \tag{204}$$

We obtain the eigenvalue equation

$$R^2 A^1 = -A^1 \,, \tag{205}$$

with the eigenvalue

$$R = \pm \mathrm{i} \,. \tag{206}$$

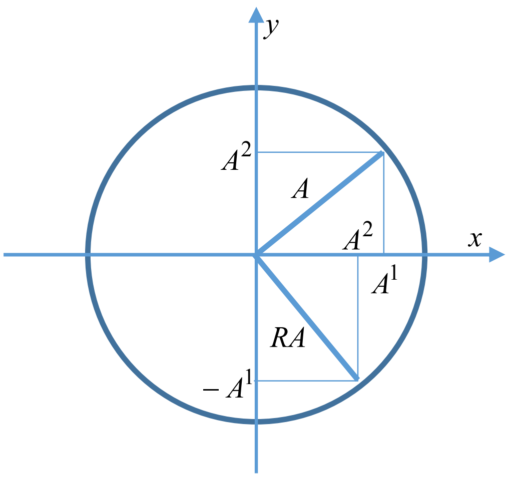


**Fig 4** Rotation with of a vector in a plane .

We define similar, symmetric expressions for the rotation of a tensor,

$$
\begin{aligned}
Ru_{11} &= u_{12} = u_{21} = \frac{1}{2}\left(u_{12}+u_{21}\right) \\
Ru_{22} &= -u_{21} = -u_{12} = -\frac{1}{2}\left(u_{12}+u_{21}\right) \\
Ru_{21} &= Ru_{12} = \frac{1}{2}\left(u_{22}-u_{11}\right).
\end{aligned}
\tag{207}
$$

From the first two equations, we obtain

$$
\begin{aligned}
R\left(u_{11}+u_{22}\right) &= 0 \\
R\left(u_{11}-u_{22}\right) &= 2u_{12}\,.
\end{aligned}
\tag{208}
$$

By applying the operator $R$ to the second equation, with the third equation (207), we obtain

$$R^2\left(u_{11}-u_{22}\right) = 2Ru_{12} = u_{22}-u_{11}\,. \tag{209}$$

Thus, for the definition (207) of this operator $R$ we reobtain the eigenvalue (206). We notice that, by a rotation $R$, the first term of the proper dynamics invariant (202), with the second equation (208), takes the form of the second term of this invariant,

$$\frac{1}{2}\left[R\left(u_{11}-u_{22}\right)\right]^2 = \frac{1}{2}\left(2u_{12}\right)^2 = 2{u_{12}}^2\,. \tag{210}$$

At the same time, the second term of the proper dynamics invariant (202), with the last equation (207), takes the form of the first term of this invariant,

$$2\left(Ru_{12}\right)^2 = 2\left[\frac{1}{2}\left(u_{22}-u_{11}\right)\right]^2 = \frac{1}{2}\left(u_{11}-u_{22}\right)^2. \tag{211}$$

This means that, by rotation, the terms of the invariant (202) transform one another. We consider the rotation operator $R^{\delta\vec{\alpha}}$ with an angle $\delta\vec{\alpha}$ of a vector $A^{\mu}\left(\vec{r}\right)$,

$$
\begin{aligned}
R^{\delta\vec{\alpha}}A^{\mu}\left(\vec{r}\right) &= A^{\mu}\left(\vec{r}+\delta\vec{\alpha}\times\vec{r}\right) \\
&= A^{\mu}\left(\vec{r}\right)+\delta\vec{\alpha}\times\vec{r}\cdot\frac{\partial}{\partial\vec{r}}A^{\mu}\left(\vec{r}\right) \\
&= A^{\mu}\left(\vec{r}\right)+\delta\vec{\alpha}\cdot\vec{r}\times\frac{\partial}{\partial\vec{r}}A^{\mu}\left(\vec{r}\right) \\
&= A^{\mu}\left(\vec{r}\right)+\mathrm{i}\delta\vec{\alpha}\vec{S}A^{\mu}\left(\vec{r}\right) \\
&= e^{\mathrm{i}\vec{S}\delta\vec{\alpha}}A^{\mu}\left(\vec{r}\right),
\end{aligned}
\tag{212}
$$

as a function of the angular momentum operator,

$$\vec{S} = -\mathrm{i}\vec{r}\times\frac{\partial}{\partial\vec{r}}\,. \tag{213}$$

This operator is of the form

$$R^{\delta\vec{\alpha}} = e^{\mathrm{i}\vec{S}\delta\vec{\alpha}}\,. \tag{214}$$

From the rotation of a vector with an angle $\pi$, which is equivalent to an inversion,

$$R^{\pi}A^{\mu}\left(\vec{r}\right) = e^{\mathrm{i}S\pi}A^{\mu}\left(\vec{r}\right) = -A^{\mu}\left(\vec{r}\right), \tag{215}$$

we obtain the angular momentum eigenvalue we call spin, $S=1$. With this eigenvalue, from the invariance of a scalar with a rotation operator (214),

$$
\begin{aligned}
\text{Scalar} = u_{\mu\nu}\left(x,y\right)A^{\mu}\left(x,y\right)B^{\nu}\left(x,y\right) &= \left[R^{\delta\vec{\alpha}}u_{\mu\nu}\left(x,y\right)\right]\left[R^{\delta\vec{\alpha}}A^{\mu}\left(x,y\right)\right]\left[R^{\delta\vec{\alpha}}B^{\nu}\left(x,y\right)\right] \\
&= \left[e^{\mathrm{i}\vec{S}_t\delta\vec{\alpha}}u_{\mu\nu}\left(x,y\right)\right]\left[e^{\mathrm{i}\vec{S}\delta\vec{\alpha}}A^{\mu}\left(x,y\right)\right]\left[e^{\mathrm{i}\vec{S}\delta\vec{\alpha}}B^{\nu}\left(x,y\right)\right] \\
&= \left[e^{\mathrm{i}\vec{S}_t\delta\vec{\alpha}}u_{\mu\nu}\left(x,y\right)\right]\left[e^{\mathrm{i}\delta\alpha}A^{\mu}\left(x,y\right)\right]\left[e^{\mathrm{i}\delta\alpha}B^{\nu}\left(x,y\right)\right],
\end{aligned}
\tag{216}
$$

we obtain the rotation operator of a tensor,

$$R^{\delta\vec{\alpha}}u_{\mu\nu}\left(x,y\right) = e^{\mathrm{i}\vec{S}_t\delta\vec{\alpha}}u_{\mu\nu}\left(x,y\right) = e^{-2\mathrm{i}\vec{S}\delta\vec{\alpha}}u_{\mu\nu}\left(x,y\right) = e^{-2\mathrm{i}\delta\alpha}u_{\mu\nu}\left(x,y\right), \tag{217}$$

which means a tensor angular momentum eigenvalue $S_t = -2$, we call the 'graviton spin'. In other words, the amplitude tensor $u_{\mu\nu}\left(x,y\right)$ describes a rotation of any intrinsic matter element as a component of a quantum particle (162), we call 'graviton', with the dynamic invariant (202). As we have shown above, the matrix elements of this tensor change such that, at any rotation with an angle $\pi/2$, a term of this invariant takes the form of the other term. This means that for this rotation, with the spin $2$, any term of the invariant $I_u$ takes two times the same value in a complete rotation $\alpha = 2\pi$. It is interesting to consider the rotation operator of a particle wave function,

$$
\begin{aligned}
R_{\delta\vec{\alpha}}\psi\left(\vec{r}\right) &= \psi\left(\vec{r}+\delta\vec{\alpha}\times\vec{r}\right) = \psi\left(\vec{r}\right)+\delta\vec{\alpha}\times\vec{r}\,\frac{\partial}{\partial\vec{r}}\psi\left(\vec{r}\right) \\
&= \psi\left(\vec{r}\right)+\delta\vec{\alpha}\cdot\underbrace{\vec{r}\times\frac{\partial}{\partial\vec{r}}}_{\mathrm{i}\vec{S}}\psi\left(\vec{r}\right),
\end{aligned}
\tag{218}
$$

which is

$$R_{\vec{\alpha}} = e^{\mathrm{i}\vec{S}\vec{\alpha}}\,. \tag{219}$$

If we consider a two-particle state

$$\psi_{i_1i_2}\left(\vec{r}_1,\vec{r}_2\right) = \left\langle\vec{r}_1,\vec{r}_2\middle|i_1,i_2\right\rangle, \tag{220}$$

with an inversion operator $I$,

$$\left\langle\vec{r}_2,\vec{r}_1\middle|i_1,i_2\right\rangle = I\left\langle\vec{r}_1,\vec{r}_2\middle|i_1,i_2\right\rangle, \tag{221}$$

by applying two times this operator,

$$\left\langle\vec{r}_1,\vec{r}_2\middle|i_1,i_2\right\rangle = I\left\langle\vec{r}_2,\vec{r}_1\middle|i_1,i_2\right\rangle = I^2\left\langle\vec{r}_1,\vec{r}_2\middle|i_1,i_2\right\rangle, \tag{222}$$

we obtain the eigenvalues

$$I^2 = 1\begin{cases} I_1 = -1 & \text{for Fermions} \\ I_2 = \ \ 1 & \text{for Bosons}. \end{cases} \tag{223}$$

Taking into account that a particle interchange can be considered as a particle double rotation with an angle $\pi$ (**Fig. 5**),

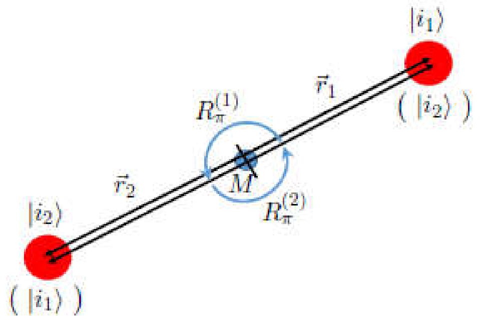


**Fig 5** Particle inversion as a double rotation.

$$R_\pi^{(1)} = R_\pi^{(2)} = e^{\mathrm{i}\pi S}, \qquad (224)$$

we obtain the relation between the inversion eigenvalue and spin, conventionally called 'the spin-statistics relation',

$$I = e^{\mathrm{i}2\pi S}\begin{cases} I_1 = R_\pi^{(1)} R_\pi^{(2)} = e^{\mathrm{i}2\pi S_1} = -1 & \Rightarrow S_1 = \frac{1}{2}, \frac{3}{2}, \ldots \quad \text{for Fermions} \\ I_2 = R_\pi^{(1)} R_\pi^{(2)} = e^{\mathrm{i}2\pi S_2} = \ \ 1 & \Rightarrow S_2 = 1, 2, \ldots \quad \text{for Bosons}. \end{cases} \qquad (225)$$

Compared to the rotation operator (214) describing the proper dynamics of the intrinsic matter as a component of a quantum particle, the rotation operator (219) describes the proper dynamics of the extrinsic matter of this particle. For the extrinsicmatter distribution of a quantum particle we define the flux

$$J^\mu = \rho \dot{x}^\mu, \qquad (226)$$

satisfying the conservation condition of a null covariant divergence,

$$J^\mu{}_{;\mu} = J^\mu_{,\mu} + \Gamma^\mu_{\nu\mu} J^\nu = J^\nu_{,\nu} + \Gamma^\mu_{\nu\mu} J^\nu = 0. \qquad (227)$$

We consider the second kind Christoffel symbol

$$\Gamma^\mu_{\nu\sigma} = g^{\mu\lambda}\Gamma_{\lambda\nu\sigma} = g^{\mu\lambda}\frac{1}{2}\left(g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\sigma\nu,\lambda}\right), \qquad (228)$$

with the index contraction $\sigma = \mu$. By taking into account the symmetry of the metric tensor, for the Christoffel symbol in equation (227) we obtain the expression

$$\begin{aligned} \Gamma^\mu_{\nu\mu} &= \frac{1}{2}\underline{\underline{g^{\mu\lambda}}}\left(\underline{g_{\lambda\nu,\mu}} + g_{\lambda\mu,\nu} - \underline{g_{\mu\nu,\lambda}}\right) = \frac{1}{2} g^{\mu\lambda} g_{\lambda\mu,\nu} \\ &= \frac{1}{2} g^{-1} g_{,\nu} = \frac{1}{2}(-g)^{-1}(-g)_{,\nu} = \frac{\left(\sqrt{-g}\right)_{,\nu}}{\sqrt{-g}}, \end{aligned} \qquad (229)$$

as a function of the determinant $g$ of the metric tensor. With this expression, the conservation condition of a null covariant divergence (227) takes the form of a null ordinary divergence,

$$J^\mu{}_{;\mu}\sqrt{-g} = \left(J^\mu\sqrt{-g}\right)_{,\mu} = 0, \qquad (230)$$

for which we can apply the Gauss integral formula. Thus, from the integral

$$\int_V \left(J^\mu \sqrt{-g}\right)_{,\mu} \mathrm{d}^3 x = 0, \qquad (231)$$

over the space coordinates, by separating the time derivative from the spatial derivatives, we obtain the conservation formula

$$\left(\int_V J^0\sqrt{-g}\,\mathrm{d}^3 x\right)_{,0} = -\int_V \left(J^m\sqrt{-g}\right)_{,m}\mathrm{d}^3 x = -\oint_{\Sigma_V} J^m\sqrt{-g}\,\mathrm{d}^2 x_m, \quad m = 1,2,3. \qquad (232)$$

For the classical case of a low velocity always considered in the proper system of a quantum particle, $\dot{x}^m << 1: \ \ g \approx -1, \ J^0 = \rho\dot{x}^0 \approx \rho, \ \ J^m = \rho\dot{x}^m$, this formula takes the form of the conventional equation of conservation,

$$\left(\int_V \rho\,\mathrm{d}^3\vec{r}\right)_{,0} = -\oint_{\Sigma_V} \vec{J}\mathrm{d}^2\vec{r}. \qquad (233)$$

Thus, a quantum particle appears as a distribution of conservative extrinsic matter, we call 'quantum matter' with a proper rotation with spin $\frac{1}{2}$ for Fermions and spin $1$ for Bosons, and with a component of intrinsic coordinates of the Universe, we call graviton, with the spin $2$.

***Summary***

We found that the conventional Schrödinger equation is contradictory to the basic Hamilton equations and to the principle of the energy conservation. A correct quantum dynamical equation has been obtainedonly when the Hamiltonian has been replaced by the Lagrangian – in this case, the Hamilton equations are obtained as group velocities of the wave packets describing the particle dynamics. With a relativistic Lagrangian, the relativistic principle of invariance of the time-space interval takes the form of a relativistic quantum principle of invariance of the time dependent phase of a quantum particle wave function. According to the general theory of relativity, we found that any acceleration in an external (non-gravitational) field is perpendicular to the velocity. In this case, the dynamics of the matter density is describable by a Fourier series expansion in waves perpendicular to the velocity, which, for a quantum particle, is normalized to the mass of the Lagrangian in the phases of these waves – QUANTUM MECHANICS. We considered black quantum particles with Lagrangians proportional to the particle masses, and visible quantum particles with additional Lagrangian terms proportional to the electric charges. For the interaction with an electromagnetic field, described by a vector potential conjugated to the coordinates and a scalar potential conjugated to time, we obtained Lorentz's force for the electromagnetic fields defined by these potentials, and the Maxwell equations for these fields. From the Lagrangian as a function of the Hamiltonian and the momentum-velocity product, we obtained a relativistic Schrödinger-type equation which, in the explicit form, yields Schrödinger-Dirac type equations with new terms depending on velocity and momentum. For an energy eigenvalue, we obtained a non-linear relativistic equation with spin interaction. In the non-relativistic case, of asmall velocity and electric potential, this equation reduces the conventional Schrödinger equation with the spin interaction. For a quantum particle in a central gravitational field, we derived the Schwarzschild solution for the metric tensor in the time dependent phase of the wave function of this particle. We obtained the velocity of a matter element which decreases when this element approaches the boundary of a black hole, tending to the null value while reaching this boundary. Thus, the boundary of a black hole cannot be passed neither from the outside, nor from the inside. However, by taking into account that any matter element is only a part of a quantum particle, asit is joint to the other parts of this particle and, in the realistic cases, to many other quantum particles, perturbing the gravitational field, we found that the boundary of a black hole is not totally unpassable: absorption and evaporation processes arise. With the Schwarzschild metric elements, from the geodesic equations we obtained the particle acceleration in a gravitational field. Compared to the Newtonian acceleration, we obtained a correction term which, for a black hole, takes the null value at the boundary of this black hole. For the metric tensor in a gravitational wave we considered two possible solutions: 1) a second order solution, proportional to an amplitude tensor and a polarization tensor, and 2) a first-order solution, proportional to an amplitude tensor and a polarization vector. From the second-order solution, we obtained an oscillation of the particle in a gravitational wave, while from the first-order solution we obtained a particle acceleration induced by a gravitational wave. From the normalization condition of the polarization vector, we obtained an invariant of the amplitude tensor including the particle acceleration and a proper dynamics of this particle. From the invariance of a scalar as a product of the amplitude tensor with two vectors with spin 1, for the intrinsic matter element as a component of a quantum particle, we call 'graviton', we obtained a rotation operator with the spin 2. At the same time, from the equivalence of a double rotation of a two particle wave function with the inversion of these particles, we found a rotation operator with a half integer spin for Fermions and an integer spin for Bosons. Essentially, we found that a quantum particle is a distribution of a quantized quantity of extrinsic matter with a mass, an integer or a half-integer spin, possible charges, and a rotation of its intrinsic component, in the time-space coordinates, we call graviton, with a spin 2.

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